Mathematical aspects of phase rotation ambiguities

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work done in collaboration with Alfred Švarc

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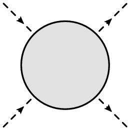
Introduction

Consider spin-less $2 \to 2$-amplitude: $A(W, \theta) =$ 

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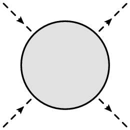
Consider spin-less $2 \to 2$-amplitude: $A(W, \theta) =$ 

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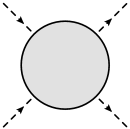
Consider spin-less $2 \to 2$-amplitude: $A(W, \theta) =$ 

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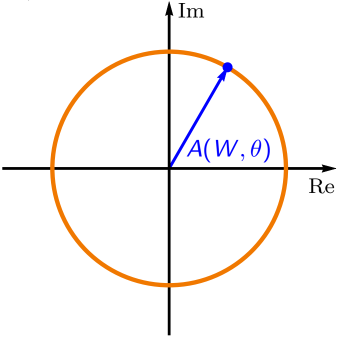

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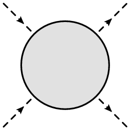
Consider spin-less $2 \to 2$-amplitude: $A(W, \theta) =$ 

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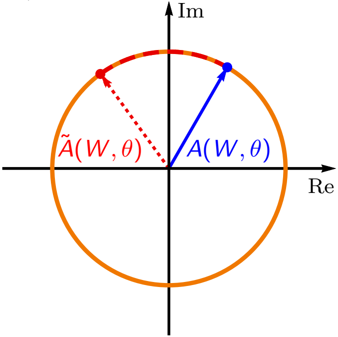

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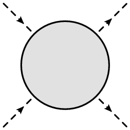
Consider spin-less $2 \to 2$-amplitude: $A(W, \theta) =$ 

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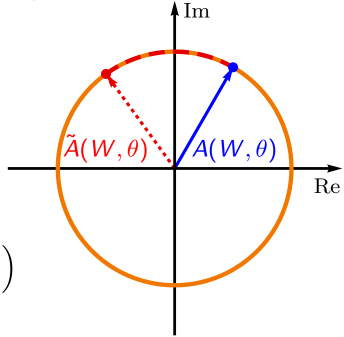
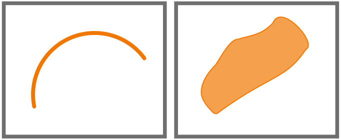


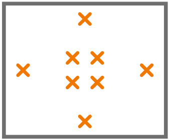
$\Rightarrow$ Implications for partial wave decomp.
$A(W, \theta) = \sum_{\ell=0}^{\infty} (2\ell + 1) A_{\ell}(W) P_{\ell}(\cos\theta)$,
$\left(\Leftrightarrow A_{\ell}(W) = \frac{1}{2} \int_{-1}^{1} d\cos\theta \, A(W, \theta) P_{\ell}(\cos\theta) \right)$
and in particular for truncated PWA?

Continuum- vs. discrete ambiguities

	Continuum ambiguities	Discrete ambiguities
*) Definition:	$\tilde{A}(W,\theta) = e^{i\Phi(W,\theta)} A(W,\theta)$	For $A(W,\theta) = \hat{A}(W,\theta)\left(\cos\theta - \alpha\right)$, conjugate the zero/root: $\alpha \rightarrow \alpha^{*}$.
	[Bowcock & Burkhardt], [L. P. Kok], ...	[A. Gersten], [E. Barrelet], [L. P. Kok], [A. S. Omelaenko], ...
*) Invariance:	$\sigma_0 = \left\| A \right\|^2 = A^{*} A$ $\rightarrow \tilde{A}^{*} \tilde{A} = e^{-i\Phi} A^{*} e^{i\Phi} A$ $= e^{i(\Phi - \Phi)} A^{*} A = A^{*} A = \sigma_0$ ✓	$\sigma_0 = \left\| \hat{A} \right\|^2 \left(\cos\theta - \alpha^{*}\right)\left(\cos\theta - \alpha\right)$ $\rightarrow \left\| \hat{A} \right\|^2 \left(\cos\theta - \left[\alpha^{*}\right]^{*}\right)\left(\cos\theta - \alpha^{*}\right)$ $= \left\| \hat{A} \right\|^2 \left(\cos\theta - \alpha^{*}\right)\left(\cos\theta - \alpha\right)$ $= \sigma_0$ ✓

*) Illustration:





Grey box: space of partial wave amplitudes $\{A_0, \ldots, A_\infty\}$, or $\{A_0, \ldots, A_L\}$.

Orange: parameter-regions of ambiguity, i.e. with same σ_0.

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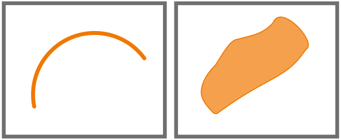
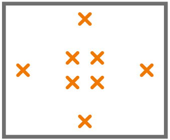
∗) Illustration:		

Now: consider only mathematical ambiguities, disregarding physical constraints (e.g. unitarity!). Are discrete and continuum ambiguities different/related?

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[J. C. Adams (1878)]

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Mixing-formula: $$\tilde{A}_{\ell}(W) = \sum_{\ell'=0}^{\infty} L_{\ell'}(W) \sum_{m=|\ell'-\ell|}^{\ell'+\ell} \left\langle \ell', 0; \ell, 0 | m, 0 \right\rangle^{2} A_{m}(W)$$

Some notes:

*) Similar mixing-phenomena discussed in literature (for spin-reactions): πN-sc.: [N. W. Dean & P. Lee (1972)] , π-photoprod.: [Omelaenko (1981)] .

*) For angle-independent phase $\Phi(W,\theta) = \Phi(W)$: $e^{i\Phi(W,\theta)} = e^{i\Phi(W)} \equiv L_0(W)$ and $\tilde{A}_{\ell}(W) = L_0(W) A_{\ell}(W) = e^{i\Phi(W)} A_{\ell}(W)$.
$\longrightarrow A_{\ell}(W)$ do not mix any more & are rotated by the same phase!

*) Non-linearity introduced by the exp-function in the rotation $e^{i\Phi(W,\theta)}$ generates complicated mixings, even when the phase $\Phi(W,\theta)$ itself is *simple*, e.g. $\Phi(W,\theta) = a(W) + b(W)\cos\theta$.

$\Rightarrow$ Ambiguity looks vicious! However, is it really that bad in practice?

$\hookrightarrow$ Get closer to answer by comparing to discrete ambiguities!

Discrete ambiguities: example I

$*$) Model truncated at the P-wave:

$A(W, \theta) = \sum_{\ell=0}^{1} (2\ell + 1) A_{\ell}(W) P_{\ell}(\cos\theta) = A_0(W) + 3\, A_1(W) \cos\theta$.

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$$\sigma_{0} = \left|A_{0} + 3\,A_{1}\cos\theta\right|^{2} = \left|A_{0}\right|^{2} + 6\mathrm{Re}\left[A_{0}^{\ast} A_{1}\right]\cos\theta + 9\left|A_{1}\right|^{2}\cos^{2}\theta.$$

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| Fix <u>phase-convention</u>: $A_1 \equiv \mathrm{Re}\left[A_1\right] > 0$

$$\Rightarrow \sigma_0 = \left|A_0\right|^2 + 6 A_1 \mathrm{Re}\left[A_0^{*}\right]\cos\theta + 9 A_1^2 \cos^2\theta \equiv c_0 + c_1\cos\theta + c_2\cos^2\theta.$$

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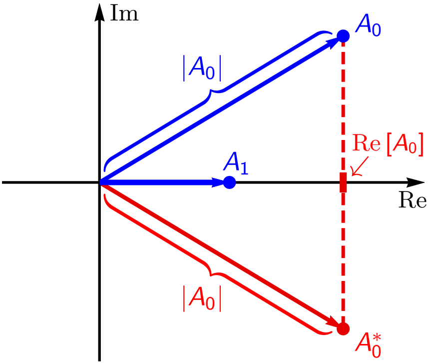
$\Rightarrow$ Constraints on partial waves:



$$\left\{c_0, c_2\right\} \Rightarrow \left\{\left|A_0\right|, A_1\right\},$$

$$\left\{c_1\right\} \Rightarrow \left\{\mathrm{Re}\left[A_0^*\right] = \mathrm{Re}\left[A_0\right]\right\},$$

From $\left|A_0\right|^2 = \mathrm{Re}\left[A_0\right]^2 + \mathrm{Im}\left[A_0\right]^2$, we get:

$$\underline{\mathrm{Im}\left[A_0\right] = \pm\sqrt{\left|A_0\right|^2 - \mathrm{Re}\left[A_0\right]^2}}.$$

Discrete ambiguities: example II

$*)$ Another point of view: linear-factor decomposition

$$A = A_0 + 3\,A_1 \cos\theta = 3A_1 \left(\cos\theta - \frac{[-A_0]}{3A_1} \right) \equiv \underline{\lambda \left(\cos\theta - \alpha_1 \right)}.$$

Phase-convention fixed via the normalization: $A_1 \equiv \mathrm{Re}\left[A_1\right] =: \underline{\lambda/3}$.

$*)$ Discrete ambiguity derived from Gersten-root α_1: [A. Gersten (1969)]

$$\underline{\alpha_1 \longrightarrow \alpha_1^*}.$$

$*)$ Cross section σ_0 easily seen to be invariant:

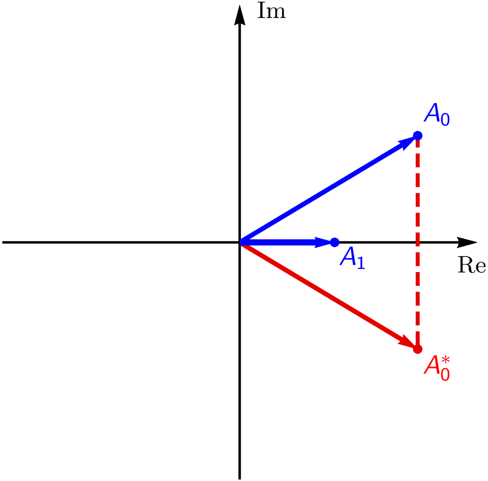
$$\begin{aligned}\sigma_0 = \left|A\right|^2 &= \left|\lambda\right|^2 \left(\cos\theta - \alpha_1^*\right)\left(\cos\theta - \alpha_1\right) \\ &\longrightarrow \left|\lambda\right|^2 \left(\cos\theta - \left[\alpha_1^*\right]^*\right)\left(\cos\theta - \alpha_1^*\right) \\ &= \left|\lambda\right|^2 \left(\cos\theta - \alpha_1^*\right)\left(\cos\theta - \alpha_1\right) = \sigma_0 \;\checkmark\end{aligned}$$

$*)$ Discrete ambiguity acting on the partial waves:

$$\alpha_1 = -\frac{A_0}{3A_1} \longrightarrow \alpha_1^* = -\frac{A_0^*}{3A_1^*} \equiv -\frac{A_0^*}{3A_1}.$$

Discrete ambiguities: example III

$\rightarrow$ Recover the same ambiguity $\underline{A_0 \rightarrow A_0^*}$ and geometric picture as before:



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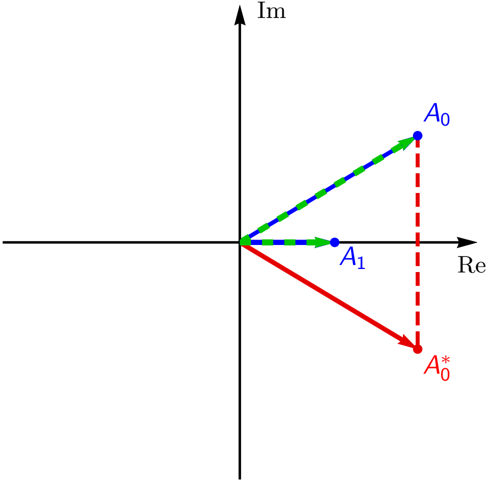

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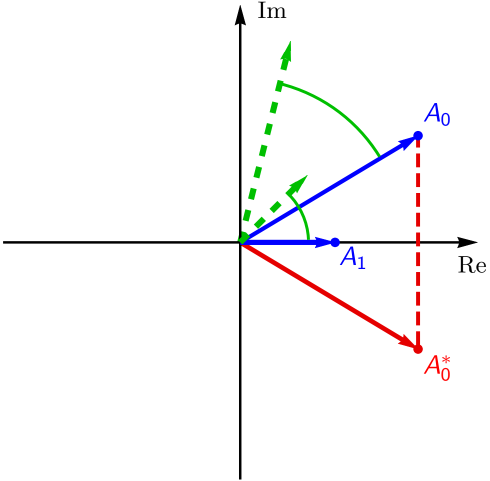

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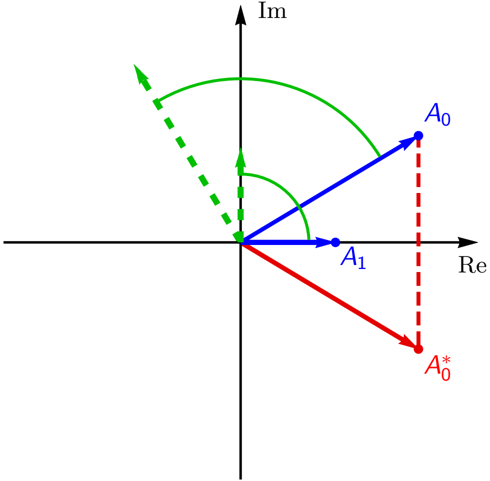

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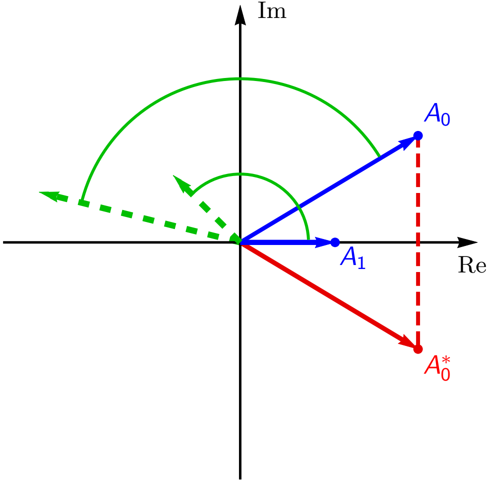

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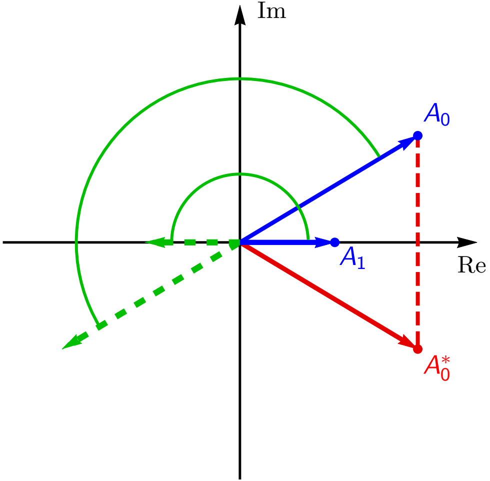

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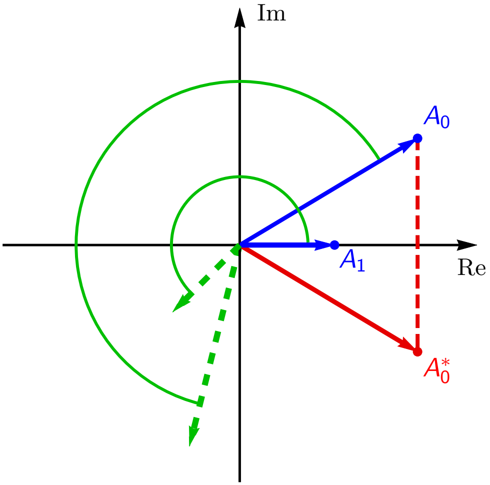

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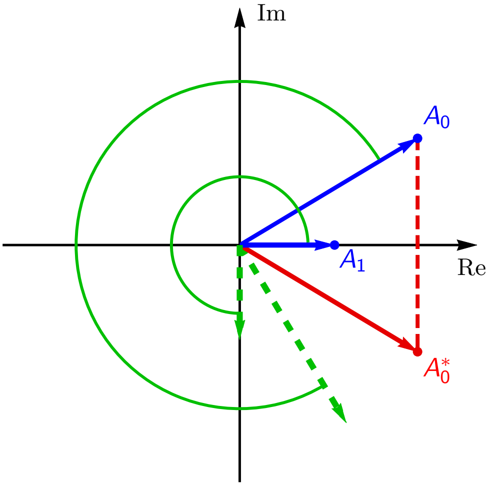

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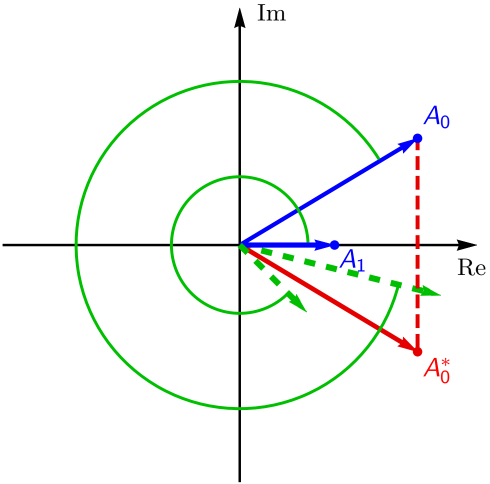

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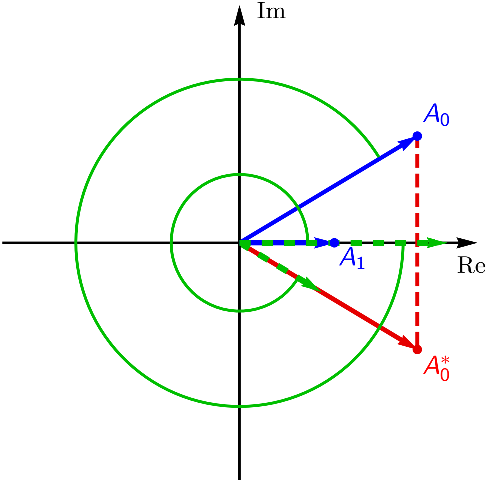

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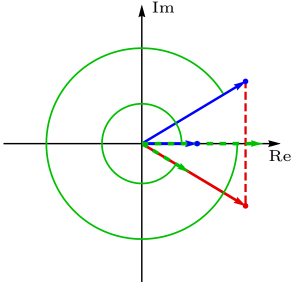

Try to rotate $\{A_0, A_1\}$ into $\{A_0^*, A_1\}$ by applying the same phase-angle to both partial waves, i.e. by performing an energy-dependent phase-rotation $\Phi(W)$.

$\hookrightarrow$ Impossible!

However: The modulus $|A(W, \theta)| = \sqrt{\sigma_0(W, \theta)}$ is left invariant by the discrete ambiguity $A_0 \rightarrow A_0^*$.

$\rightarrow$ Transformation can (effectively) only be a rotation.

$\rightarrow$ It has to be an angle-dependent rotation!

How to generalize these results to higher $L = \ell_{\mathrm{max}}$?

Discrete ambiguities: general formalism I

$*$) A general truncated (i.e. polynomial-) amplitude for arbitrary $L = \ell_{\mathrm{max}}$, $A = \sum_{\ell=0}^{L} (2\ell + 1) A_{\ell} P_{\ell}(\cos\theta)$, has the linear-factor decomposition:

$$A = \lambda \left(\cos\theta - \alpha_1\right) \left(\cos\theta - \alpha_2\right) \ldots \left(\cos\theta - \alpha_L\right) \text{ , with } \lambda \propto A_L .$$

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- $\ast$) There exist 2^L solutions that leave the differential cross section

$$\sigma_0 = \left|A\right|^2 = \left|\lambda\right|^2 \prod_{i=1}^{L} \left(\cos\theta - \alpha_i^{\ast}\right) \left(\cos\theta - \alpha_i\right) \text{ invariant.}$$

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*) Example: truncation at $L = 2$, i.e. S-, P- and D-waves $\{ A_0, A_1, A_2 \}$.
Then, there are 2 roots $\{ \alpha_1, \alpha_2 \}$ and $2^2 = 4$ ambiguities:

$$\left\{ \boldsymbol{\pi}_0 \left(\alpha_1 \right), \boldsymbol{\pi}_0 \left(\alpha_2 \right) \right\} = \left\{ \alpha_1, \alpha_2 \right\} \quad , \quad \left\{ \boldsymbol{\pi}_1 \left(\alpha_1 \right), \boldsymbol{\pi}_1 \left(\alpha_2 \right) \right\} = \left\{ \alpha_1^{\ast}, \alpha_2 \right\} ,$$

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Discrete ambiguities: general formalism II

$*)$ From the original truncated model $A = \lambda \prod_{i=1}^{L} \left(\cos\theta - \alpha_i \right)$, one can transform to 2^L ambiguous amplitudes, i.e. for $n = 0, \ldots, (2^L - 1)$:

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"*For fixed L these* (i.e. the $\boldsymbol{\pi}_n$-maps) *are the only ambiguities in determining the phase shifts from* $(d\sigma/d\Omega)$ (i.e. σ_0)."

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$$\alpha_i \rightarrow \boldsymbol{\zeta}_n \left(\alpha_i \right),$$

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$\Rightarrow$ We believe Gersten's statement! The reason why should become more clear by comparing discrete to continuum ambiguities.

Discrete ambiguities & angle-dependent phase rotations

$*)$ Naively assumed equivalence:

$\Phi(W,\theta) = \Phi(W)$, phase only energy-dependent.	$\overset{?}{\longleftrightarrow}$	$A(W,\theta)$ truncated at L $\downarrow$ $e^{i\Phi(W,\theta)}A(W,\theta)$ truncated at L.

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$\Rightarrow$ Instead: discrete ambiguities *are* angle-dependent rotations, for certain phases $\Phi_n(W,\theta)$, $n = 0, \ldots, (2^L - 1)$:

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∗) Amazing: Even though $e^{i\Phi(W,\theta)}$ is highly non-linear, one can rotate truncated models again into truncated ones!

⇒ Moreover: The $\Phi_n(W,\theta)$ are the only angle-dependent phases which leave the truncation order L un-touched!

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$\hookrightarrow$ Can this be substantiated *without* using the Gersten-formalism?

Functional methods and Conclusions

For fixed W and angular variable $x = \cos\theta$, start with an amplitude $A(W, \theta) = A(\theta) \equiv A(x)$ truncated at some L.

$\rightarrow$ Search (numerically) for functions $F(x)$ satisfying the 2 requirements:

(i) Unimodularity: $|F(x)|^2 = 1$, $\forall x \in [-1, 1]$,

(ii) The rotated model $\tilde{A}(x) = e^{i\Phi(x)} A(x) \equiv F(x) A(x)$ is truncated at L:

$$\tilde{A}_{L+k} = \frac{1}{2} \int_{-1}^{+1} dx \, F(x) A(x) P_{L+k}(x) \equiv 0, \ \forall k = 1, \ldots, \infty.$$

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In practice: Implement condition (i) on a grid $\{x_n\} \in [-1,1]$ & truncate condition (ii) at some (large) value $K = k_{\mathrm{max}}$.

$\Rightarrow$ The functions $F_n(x) = e^{i\Phi_n(x)}$, known from the Gersten-formalism, are the only solutions returned by this numerical procedure!

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Conclusion: The discrete ambiguities generated by $F_n(x) = e^{i\Phi_n(x)}$ are the remnants of the full continuum ambiguity $A(x) \rightarrow e^{i\Phi(x)}A(x)$, once the latter is restricted to truncated models!

Some references

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Thank You!

Additional Slides

Functional problem: definition

Reminder: 2 requirements demanded for the rotation function $F(x)$ which generates $\tilde{A}(x) = F(x)A(x)$:

(i) $\left|F(x)\right|^2 = 1, \forall x \in [-1,1]$ & (ii) $\frac{1}{2}\int_{-1}^{+1} dx\, F(x)A(x)P_{L+k}(x) \equiv 0, \, \forall k \geq 1.$

Define minimization-functional in a suitable way:

$$\mathbf{W}\left[F(x)\right] := \sum_{x} \left(\mathrm{Re}\left[F(x)\right]^2 + \mathrm{Im}\left[F(x)\right]^2 - 1 \right)^2 + \mathrm{Im}\left[\frac{1}{2}\int_{-1}^{+1} dx\, F(x)A(x) \right]^2$$

$$+ \sum_{k \geq 1} \left\{ \mathrm{Re}\left[\frac{1}{2}\int_{-1}^{+1} dx\, F(x)A(x)P_{L+k}(x) \right]^2 + \mathrm{Im}\left[\frac{1}{2}\int_{-1}^{+1} dx\, F(x)A(x)P_{L+k}(x) \right]^2 \right\},$$

and find phase-rotation functions that minimize of this functional:

$$\mathbf{W}\left[F(x)\right] \longrightarrow \mathrm{min.} \equiv 0,$$

$$\mathrm{for}\ F(x) \longrightarrow F_n(x),\, n = 0, \ldots, (2^L - 1).$$

Functional problem: solution Ansatz

$*)$ Discretize the interval $[-1, 1]$ into N_I equidistant points $\{x_n\}$ via:

$$\Delta x \equiv \frac{1-(-1)}{N_I} = \frac{2}{N_I};\ x_n := -1 + \left(\frac{1+2(n-1)}{2}\right)\Delta x,\ \forall n = 1, \ldots, N_I.$$

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$$F(\{y_{\ell'}, w_{\ell'}\})(x) := \sum_{\ell'=0}^{\mathcal{L}_{\mathrm{cut}}} (y_{\ell'} + i w_{\ell'}) P_{\ell'}(x).$$

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$$F(\{y_{\ell'}, w_{\ell'}\})(x) := \sum_{\ell'=0}^{\mathcal{L}_{\mathrm{cut}}} (y_{\ell'} + i w_{\ell'}) P_{\ell'}(x).$$

$y_{\ell'} = \mathrm{Re}\left[L_{\ell'}\right]$ and $w_{\ell'} = \mathrm{Im}\left[L_{\ell'}\right]$ are parameters for which to solve.

*) Use mixing-formula, for input-model $A(x) = \sum_{\ell \leq L} (2\ell + 1) A_\ell P_\ell(x)$ truncated at L:

$$\tilde{A}_{L+k}(\{y_{\ell'}, w_{\ell'}\}) = \sum_{\ell'=k}^{\min(2L+k, \mathcal{L}_{\mathrm{cut}})} (y_{\ell'} + i w_{\ell'}) \sum_{m=|L+k-\ell'|}^{L} \left\langle \ell', 0; \ell, 0 | m, 0 \right\rangle^2 A_m,$$

$$\forall\, k = 1, \ldots, K \Rightarrow \underline{K \equiv \mathcal{L}_{\mathrm{cut}}}.$$

Functional problem: solution Ansatz

- $*)$ Discr.: $\Delta x \equiv \frac{2}{N_I}$; $x_n := -1 + \left(\frac{1+2(n-1)}{2} \right) \Delta x$, $\forall n = 1, \ldots, N_I$.
- $*)$ Legendre-expansion: $F(\{y_{\ell'}, w_{\ell'}\})(x) := \sum_{\ell' \leq \mathcal{L}_{\mathrm{cut}}} (y_{\ell'} + i w_{\ell'}) P_{\ell'}(x)$
- $*)$ $\tilde{A}_{L+k}(\{y_{\ell'}, w_{\ell'}\}) = \sum_{\ell'=k}^{\min(2L+k, \mathcal{L}_{\mathrm{cut}})} (y_{\ell'} + i w_{\ell'}) \sum_{m=|L+k-\ell'|}^{L} \left\langle \ell', 0; \ell, 0 | m, 0 \right\rangle^2 A_m$,
  $\forall\, k = 1, \ldots, K \Rightarrow \underline{K \equiv \mathcal{L}_{\mathrm{cut}}}$.

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- $\ast)$ Discr.: $\Delta x \equiv \frac{2}{N_I}$; $x_n := -1 + \left(\frac{1+2(n-1)}{2} \right) \Delta x$, $\forall n = 1, \ldots, N_I$.
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  $\forall\, k = 1, \ldots, K \Rightarrow \underline{K \equiv \mathcal{L}_{\mathrm{cut}}}$.

Minimize the quantity

$$\mathbf{W}_{\mathcal{L}} \left(\{ y_{\ell'}, w_{\ell'} \} \right) := \sum_{\{x_n\}} \left(\mathrm{Re} \left[F(\{y_{\ell'}, w_{\ell'}\})(x_n) \right]^2 + \mathrm{Im} \left[F(\{y_{\ell'}, w_{\ell'}\})(x_n) \right]^2 - 1 \right)^2$$

$$+ \mathrm{Im} \left[\tilde{A}_0(\{y_{\ell'}, w_{\ell'}\}) \right]^2 +$$

$$\sum_{k=1}^{K} \left(\mathrm{Re} \left[\tilde{A}_{L+k}(\{y_{\ell'}, w_{\ell'}\}) \right]^2 + \mathrm{Im} \left[\tilde{A}_{L+k}(\{y_{\ell'}, w_{\ell'}\}) \right]^2 \right),$$

starting from <u>randomly chosen</u> initial parameters

$$\left(y_{\ell'}^{(0)} \right)_j \in [-1, 1], \ \left(w_{\ell'}^{(0)} \right)_j \in [-1, 1], \ j = 1, \ldots, N_{\mathrm{MonteCarlo}}.$$

Functional problem for a toy model

$*)$ Run codes using the following typical (suitable) parameter values:

- Number of grid-points $|\{x_n\}|$: $N_I = 400$.
- Truncation-order in the Legendre-expansion of $F(x)$:

$$\mathcal{L}_{\mathrm{cut}} = 60, \ldots, 100$$

$\equiv K =$ number of higher p.w.'s on whom requirement (ii) is imposed.

- Number of initial conditions: $N_{\mathrm{MonteCarlo}} = 50, \ldots, 100$.

$*)$ Try a toy model truncated at the D-waves, i.e. $L = 2$:

$$A(x) = \sum_{\ell \leq 2} (2\ell + 1) A_\ell P_\ell(x) = A_0 + 3A_1 P_1(x) + 5A_2 P_2(x)$$

$$= (5 + 4i) + 3(3 + 2i) P_1(x) + 5(1 + 0i) P_2(x).$$

Functional problem: results

Omelaenko's warning about angle-dependent phase

The large amount of experimental information which is needed for the complete experiment does not allow one, however, to obtain values of partial amplitudes free from model assumptions. In fact, in a complete experiment the amplitudes F_i are determined with accuracy to the transformation

$$F_i(E_\gamma, \theta) \rightarrow \exp\left(i\varphi(E_\gamma, \theta)\right) F_i(E_\gamma, \theta),$$

!!

where $\varphi(E_\gamma, \theta)$ is an independent real function. By choosing $\varphi(E_\gamma, \theta)$ one can vary the angular distributions of the amplitudes F_i, although the observables remain unchanged. Going over then to multipole expansions, one obtains as a result various sets of partial amplitudes differing both in the number of excited waves and in their magnitudes.

Killer Argumen

In a multipole analysis with $l \leq L$ the uncertainty in the phase manifests itself as an ambiguity in the choice of L. In the amplitude corresponding to the solution with some L one can also introduce a phase depending arbitrarily on angle, and the number of terms in the multipole expansions then changes. Having this in mind, obviously it is expedient to use the smallest value L for which one achieves a description of the experimental data.

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Warning written on [Omelaenko (1981), page 6]