Complete Experiments in pseudoscalar meson photoproduction

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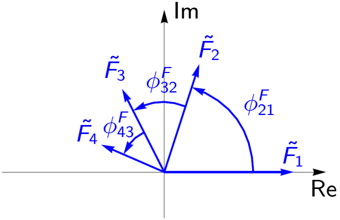

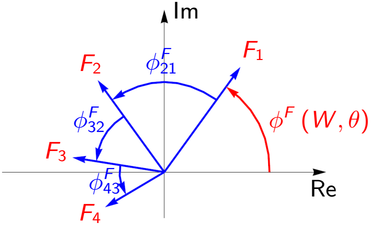

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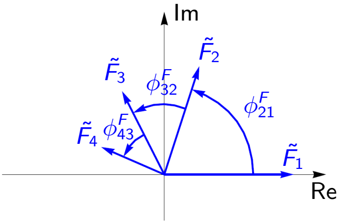

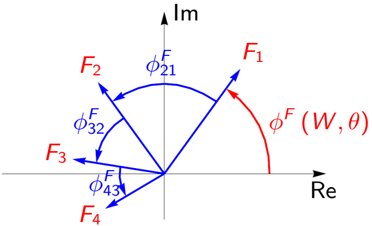

III. $\phi^F(W,\theta)$ denies access to partial waves upon extraction of the F_i.
$\longrightarrow$ Study Complete Experiments in a Truncated Partial Wave Analysis [A. S. Omelaenko (1981)] & [V. F. Grushin (1989)].

Definition of the TPWA problem

Desirable for low-energy processes: Truncate the partial wave expansion of the full spin amplitudes at some finite ℓ_{max}, e.g.

$$F_1(W,\theta) = \sum_{\ell=0}^{\ell_{\mathrm{max}}} \Big\{ \left[\ell M_{\ell+} + E_{\ell+}\right] P^{'}_{\ell+1}\left(\cos\theta\right) + \left[\left(\ell+1\right) M_{\ell-} + E_{\ell-}\right] P^{'}_{\ell-1}\left(\cos\theta\right) \Big\},$$

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Truncated Partial Wave Analysis

$$\check{\Omega}^{\alpha}(W,\theta) = \sin^{\beta_{\alpha}}\theta \left[a_0^{\alpha}(W) + a_1^{\alpha}(W)\cos\theta + a_2^{\alpha}(W)\cos^2\theta + \ldots\right]$$

$$= \sin^{\beta_{\alpha}}\theta \sum_{k=0}^{2\ell_{\mathrm{max}}+\gamma_{\alpha}} a_k^{\alpha}(W)\cos^k\theta,$$

$$a_k^{\alpha}(W) = \left\langle \mathcal{M}(W)\right| C_k^{\alpha} \left|\mathcal{M}(W)\right\rangle, \ \left|\mathcal{M}(W)\right\rangle = \left(E_{\ell\pm}(W), M_{\ell\pm}(W)\right)^T.$$

$\longrightarrow$ How many and which observables have to be measured in order to uniquely solve for the multipoles $\left\{E_{\ell\pm}(W), M_{\ell\pm}(W)\right\}$?

Counting real degrees of freedom I

- The maximal $\cos\theta$ powers in the CGLN amplitudes are:
  $F_1 \sim (\cos\theta)^{\ell_{\mathrm{max}}}$, $F_2 \sim (\cos\theta)^{\ell_{\mathrm{max}}-1}$, $F_3 \sim (\cos\theta)^{\ell_{\mathrm{max}}-1}$, $F_4 \sim (\cos\theta)^{\ell_{\mathrm{max}}-2}$.

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$\longrightarrow$ Example: differential cross section σ_0

$$\sigma_0 = \mathrm{Re}\Big[\left|F_1\right|^2 + \left|F_2\right|^2 - 2\cos(\theta) F_1^* F_2 + \frac{1}{2}\sin^2(\theta)\Big\{ \left|F_3\right|^2 + \left|F_4\right|^2 + 2F_1^* F_4 + 2F_2^* F_3 + 2\cos(\theta) F_3^* F_4 \Big\}\Big].$$

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Count maximal $\cos\theta$ powers for group S and $\mathcal{BT}$ observables:

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Counting real degrees of freedom II

Add $+1$ for $(\cos\theta)^0$-term in order to obtain:

Number of angular fit coefficients a_k^{α} provided by group S and $\mathcal{BT}$:

$\sigma_0 \sim (2\ell_{\mathrm{max}} + 1)$ $\quad \check{\Sigma} \sim (2\ell_{\mathrm{max}} - 1)$ $\quad \check{T} \sim 2\ell_{\mathrm{max}}$ $\quad \check{P} \sim 2\ell_{\mathrm{max}}$

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- $\left\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\right\}$: $8\ell_{\mathrm{max}}$ $[a_k^\alpha] > (8\ell_{\mathrm{max}} - 1)$, however: discrete ambiguities!

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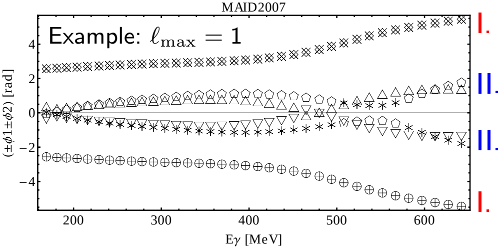
$\longrightarrow$ Comparison of real degrees of freedom seems promising!

Complete sets of observables

Study of the theoretical ambiguities of the group S observables $\left\{ \left(\frac{d\sigma}{d\Omega} \right)_0, \Sigma, P, T \right\}$ according to [A. S. Omelaenko (1981)] (see also [Wunderlich/Beck/Tiator (2014)])



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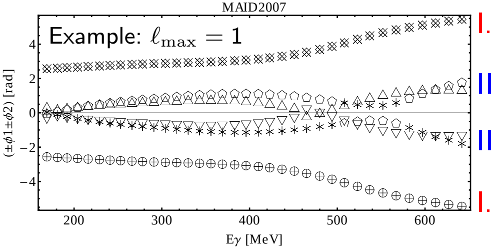

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$\longrightarrow$ Double polarization observables capable of resolving the ambiguities:
$\mathcal{BT}$: $\{F, G\}$, $\mathcal{BR}$: $\{O_{x'}, O_{z'}, C_{x'}, C_{z'}\}$, $\mathcal{TR}$: $\{T_{x'}, T_{z'}, L_{x'}, L_{z'}\}$

$\longrightarrow$ Examples of complete sets: $\underline{\{\sigma_0, \Sigma, T, P, F\}}$ or $\{\sigma_0, \Sigma, T, P, G\}$

$\longrightarrow$ Can these predictions be verified using numerical TPWA fits?

Details on the multipole Fit procedure I

Two step method:

1. Fit the angular distributions of observables, parametrized by

$$\check{\Omega}^{\alpha}\left(W,\theta\right) = \frac{q}{k} \sum_{k=\beta_{\alpha}}^{2\ell_{\mathrm{max}}+\beta_{\alpha}+\gamma_{\alpha}} \left(a_{L}\right)_{k}^{\alpha}\left(W\right) P_{k}^{\beta_{\alpha}}\left(\cos\theta\right)$$

$\Rightarrow$ Angular fit parameters $\left(a_{L}^{\mathrm{Fit}}\right)_{k}^{\alpha}$ & errors $\Delta\left(a_{L}^{\mathrm{Fit}}\right)_{k}^{\alpha}$

- Absorb $\sin^{\beta_{\alpha}}\theta$ factors into the fitting functions $P_{k}^{\beta_{\alpha}}\left(\cos\theta\right)$
- $P_{k}^{\beta_{\alpha}}\left(\cos\theta\right)$ have the advantage of being orthogonal for $\cos\theta \in [-1,1]$

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2. Minimize the functional:

$$\Phi_{\mathcal{M}}\left(\mathcal{M}_{\ell}\right) = \frac{1}{N_{F.P.}-N_{V.M.}} \sum_{\alpha,k} \left(\frac{\left(\left(a_L^{\mathrm{Fit}}\right)_k^{\alpha} - \langle\mathcal{M}_{\ell}|\left(C_L\right)_k^{\alpha}|\mathcal{M}_{\ell}\rangle\right)}{\Delta\left(a_L^{\mathrm{Fit}}\right)_k^{\alpha}}\right)^2$$

using the MATHEMATICA method

$$\mathrm{FindMinimum}\left[\Phi_{\mathcal{M}}\left(\mathcal{M}_{\ell}\right), \left\{\left\{\mathrm{Re}\left[E_{0+}\right], \left(x_1\right)_0\right\}, \ldots, \left\{\mathrm{Im}\left[M_{\ell_{\mathrm{max}}-}\right], \left(y_n\right)_0\right\}\right\}\right]$$

and varying the real and imaginary parts of the (possibly phase constrained) multipoles in the fit.

Details on the multipole fit procedure II

Question: How to choose the start parameters $\{(x_1)_0, \ldots, (y_n)_0\}$?

Ansatz: Use the total cross section $\sigma(W)$. Example: $\ell \leq \ell_{\mathrm{max}} = 1$, phase constraint $\mathrm{Im}\left[\tilde{E}_{0+}\right] = 0$ & $\mathrm{Re}\left[\tilde{E}_{0+}\right] > 0$:

$$\sigma(W) \approx 4\pi \frac{q}{k}\Big(\mathrm{Re}\left[\tilde{E}_{0+}\right]^2 + 6\,\mathrm{Re}\left[\tilde{E}_{1+}\right]^2 + 6\,\mathrm{Im}\left[\tilde{E}_{1+}\right]^2 + 2\,\mathrm{Re}\left[\tilde{M}_{1+}\right]^2 + 2\,\mathrm{Im}\left[\tilde{M}_{1+}\right]^2 + \mathrm{Re}\left[\tilde{M}_{1-}\right]^2 + \mathrm{Im}\left[\tilde{M}_{1-}\right]^2\Big)$$

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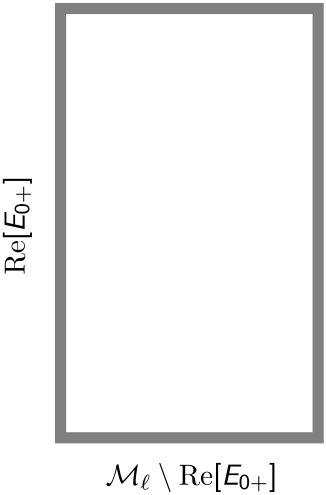
- $\sigma(W)$ constrains the intervals of the multipoles:

$$\mathrm{Re}\left[\tilde{E}_{0+}\right] \in \left[0, \sqrt{\tfrac{k}{q}\tfrac{\sigma(W)}{4\pi}}\right], \ldots, \mathrm{Im}\left[\tilde{M}_{1-}\right] \in \left[-\sqrt{\tfrac{k}{q}\tfrac{\sigma(W)}{4\pi}}, \sqrt{\tfrac{k}{q}\tfrac{\sigma(W)}{4\pi}}\right]$$

- The total cross section, being quadratic form in the multipoles, also defines an ellipsoid in the multipole space.

Generation of start values for FindMinimum

1. The total cross section $\sigma(W)$ constrains the $(8\ell_{\max} - 1)$-dimensional multipole space $\mathcal{M}_\ell$.



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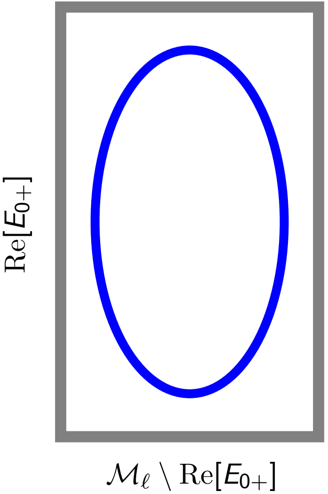

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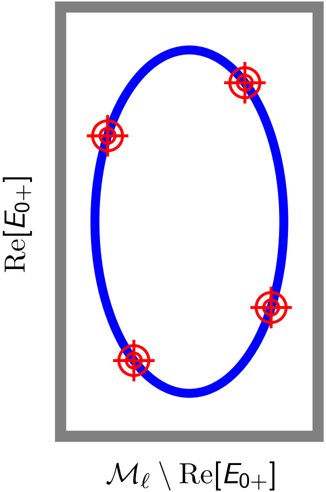
3. Solutions to the TPWA problem lie on the ellipsoid defined by $\sigma(W)$.



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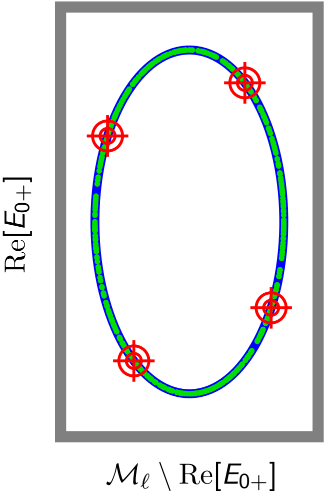
3. Solutions to the TPWA problem lie on the ellipsoid defined by $\sigma(W)$.
4. The start values for the $\mathrm{FindMinimum}$-Fit are chosen randomly on the $\sigma(W)$-ellipsoid.

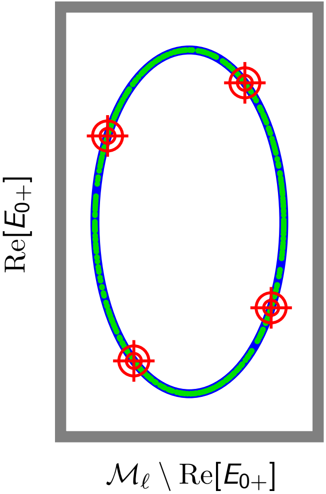
   $\Rightarrow$ Monte Carlo sampling of the multipole space.



Generation of start values for FindMinimum

5. A FindMinimum-minimization is performed for each of the randomly generated start configurations.

   $\Rightarrow N_{MC} = \#$ of M.C. start configurations
   $= \#$ of (possibly redundant) solutions



Cut selections for solution "data" I

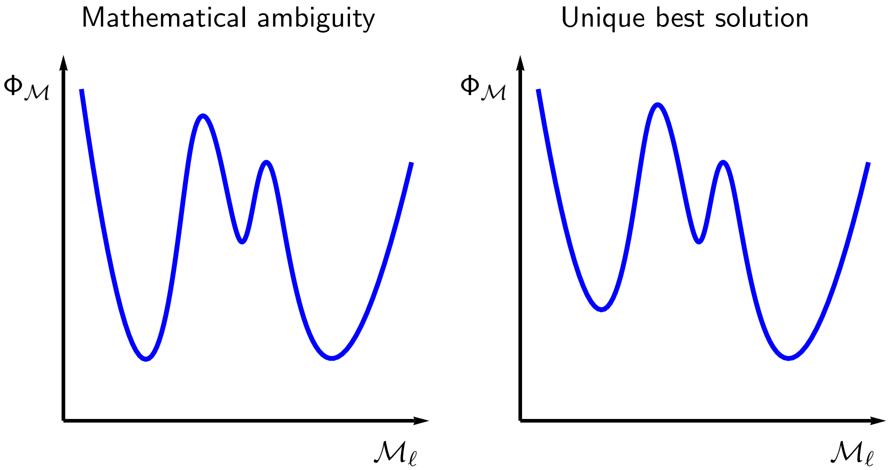
- Loop through N_{MC} Levenberg Marquardt solutions for each energy bin.
- Determine solution with the best $\Phi_{\mathcal{M}}$, i.e. $\Phi_{\mathcal{M}}^{\mathrm{best}}$ as well as the corresponding multipoles $\mathcal{M}_{\ell}^{\mathrm{best}}$.
- It is also reasonable to apply "cut selections" to all obtained LM-solutions $\Phi_{\mathcal{M}}^{j}$ according to

$$\frac{\Phi_{\mathcal{M}}^{j} - \Phi_{\mathcal{M}}^{\mathrm{best}}}{\Phi_{\mathcal{M}}^{\mathrm{best}}} < \epsilon .$$

- Examples: $\epsilon = 1$ for solutions vaguely compatible in $\Phi_{\mathcal{M}}$, or adjust ϵ to the numerical fit precision for mathematically equivalent solutions.

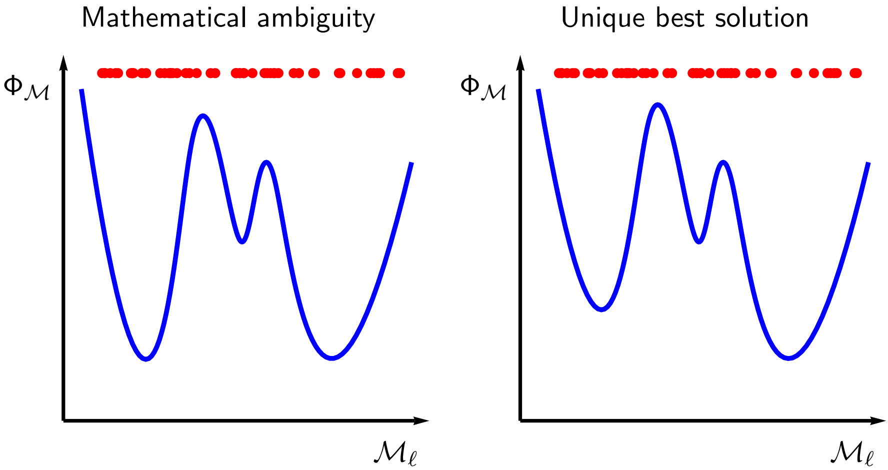
$\longrightarrow$ Store the solutions passing the cuts, later: used for histograms

Cut selections for solution "data" II



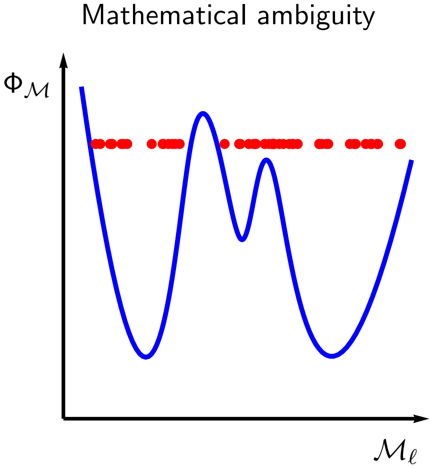
The $\Phi_{\mathcal{M}}$ is defined by the fitted Legendre coefficients $\left(a_L^{\mathrm{Fit}}\right)_k^{\alpha}$.

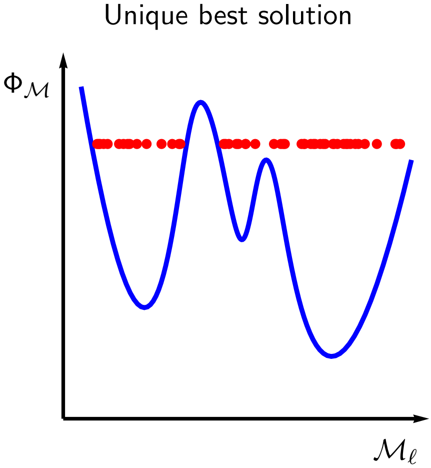
Cut selections for solution "data" II



Start values have been distributed on the relevant part of the space $\mathcal{M}_{\ell}$.

Cut selections for solution "data" II

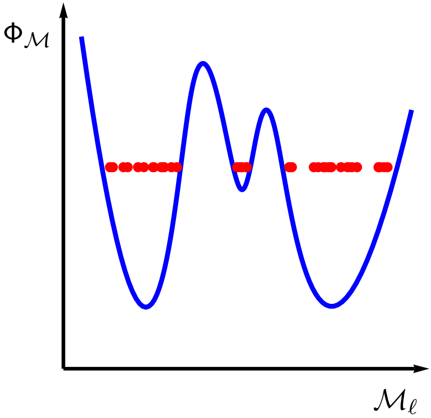




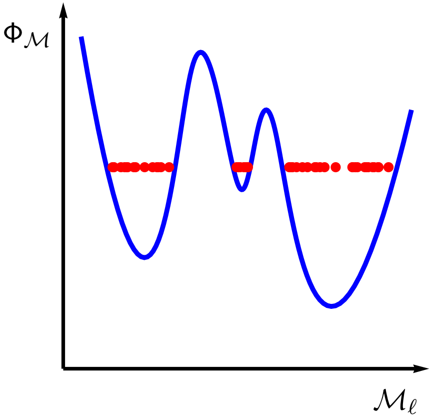
Minimizations of $\Phi_{\mathcal{M}}$ converge within several iterations.

Cut selections for solution “data” II

Mathematical ambiguity



Unique best solution



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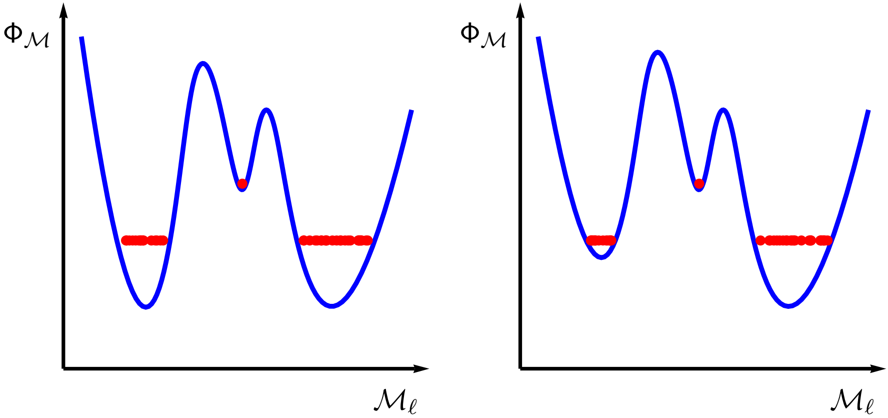

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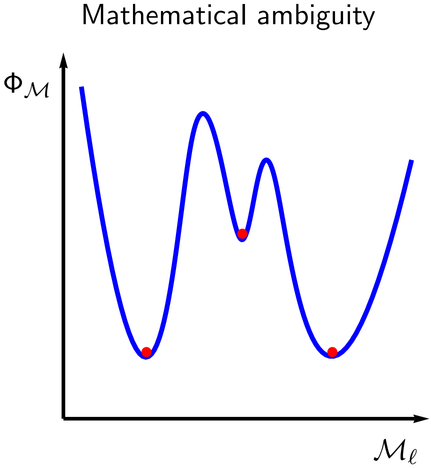

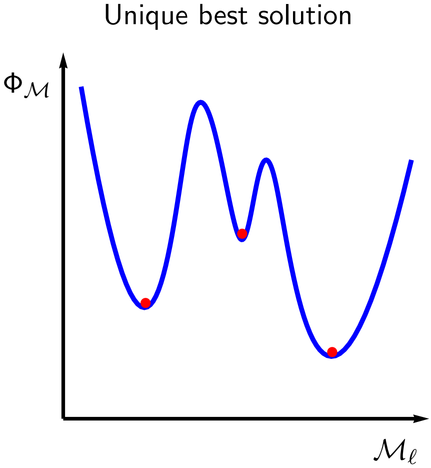

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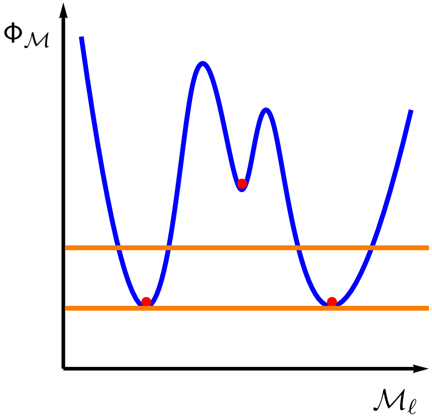

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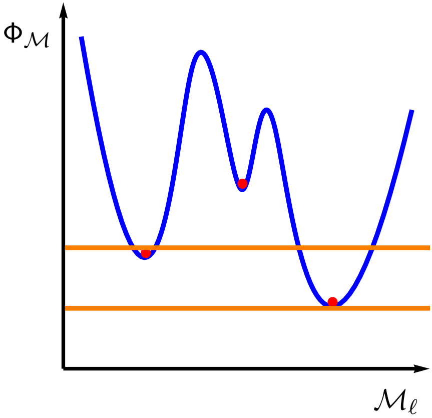
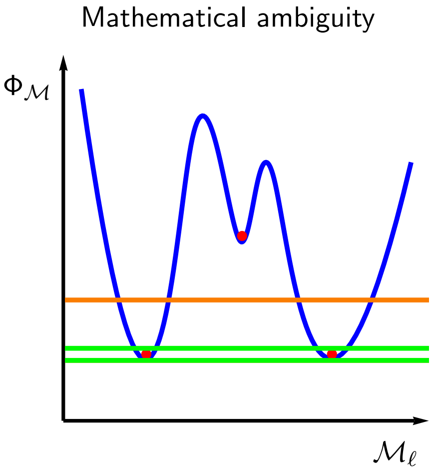


Cut selection using $\epsilon = 1$

Cut selections for solution "data" II



Cut selection using $\epsilon = 1$ / Cut selection using $\epsilon \sim \mathrm{num.\,precision}$

Fitted datasets

The following datasets were investigated in the energy region $E_{\gamma}^{\mathrm{LAB}} = 300 \ldots 350$ MeV for the process $\gamma p \rightarrow \pi^0 p$:

I. Data taken at the MAMI facility:
   - σ_0: 20 energy points for $E_{\gamma}^{\mathrm{LAB}} \in [302.010, 348.280]$ MeV
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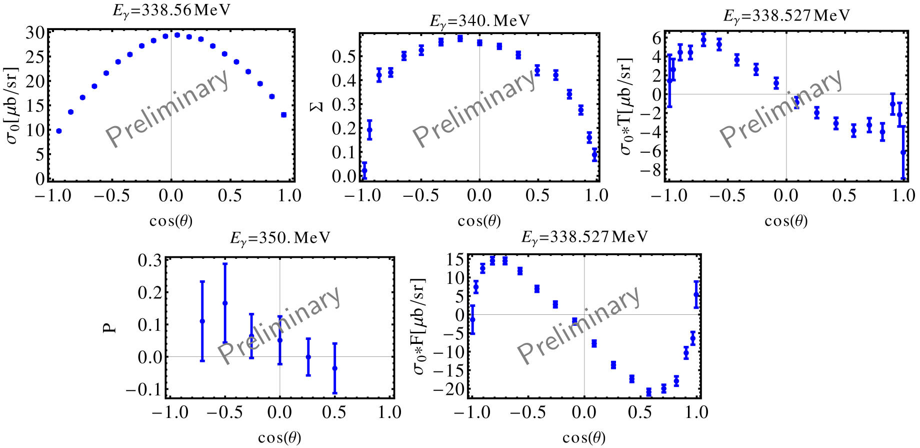
$\rightarrow$ 2214 data points taken into account for $\gamma p \rightarrow \pi^0 p$.

TPWA fits using the bootstrapping method

$$E_\gamma^{\mathrm{LAB}} \simeq E_\gamma^{\Delta} \simeq 338 \text{ MeV}$$



Angular distributions of data as provided are shown.

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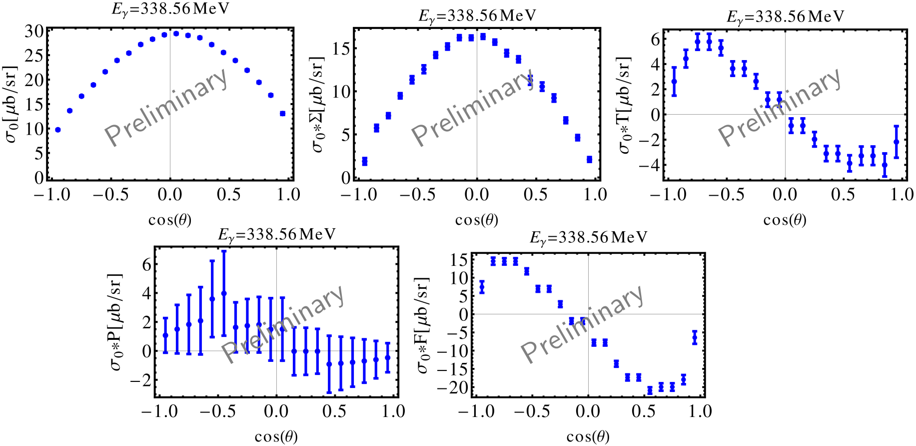

The data are re-binned to the kinematic grid dictated by the σ_0 measurement. Profile functions are calculated.

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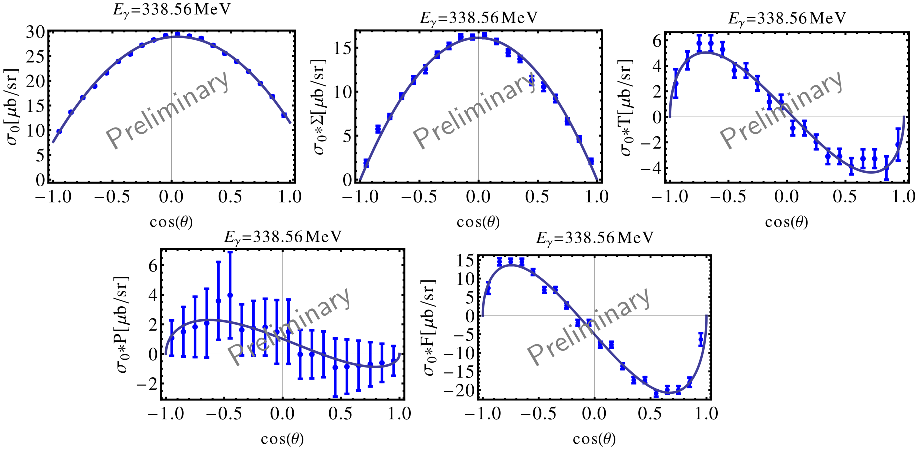

Profile functions for the original dataset are fitted with an S- and P-wave truncation ($\ell_{\max} = 1$).

TPWA fits using the bootstrapping method

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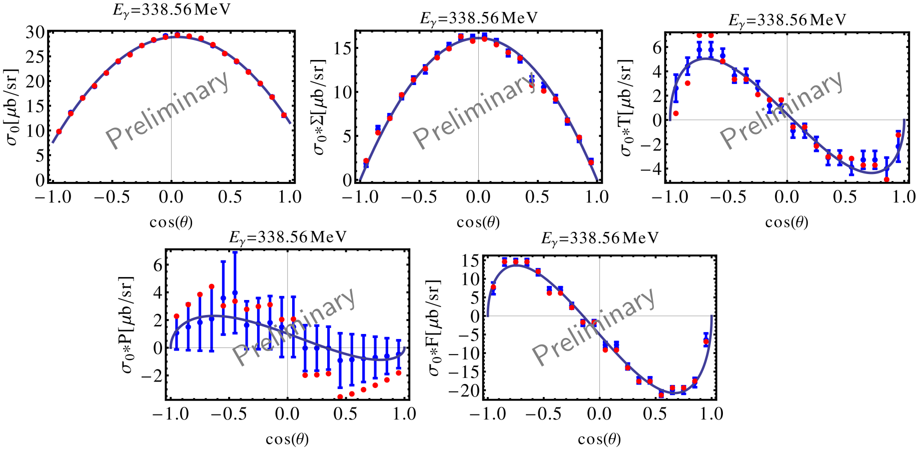
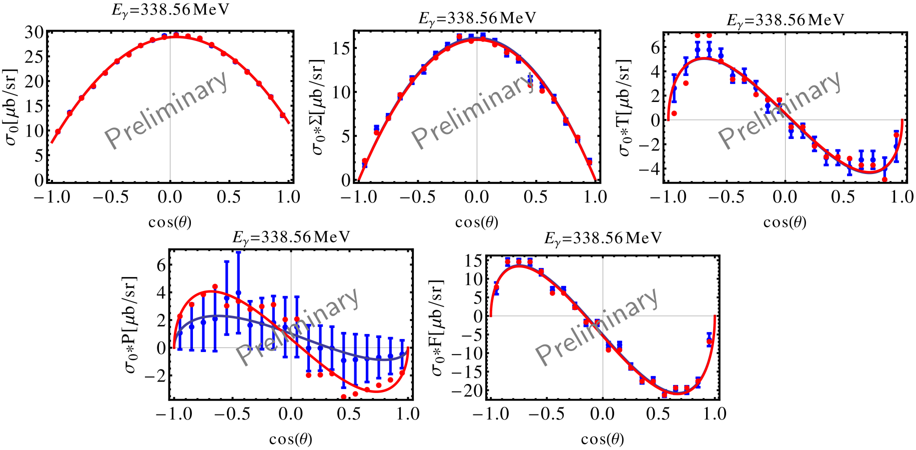


Generate 1 additional dataset using gaussian PDFs
(cf. [A. Sandorfi, Trento, 2014]).

TPWA fits using the bootstrapping method

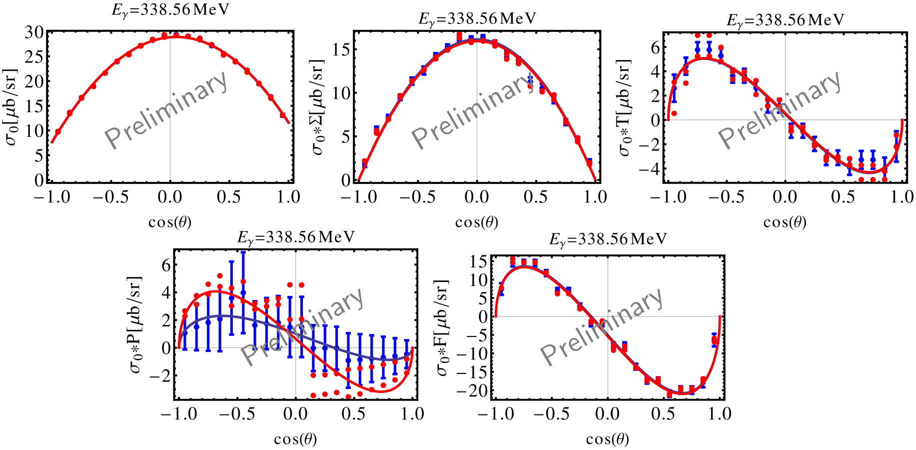
$$E_\gamma^{\mathrm{LAB}} \simeq E_\gamma^{\Delta} \simeq 338 \text{ MeV}$$



Fit the additional dataset.

TPWA fits using the bootstrapping method

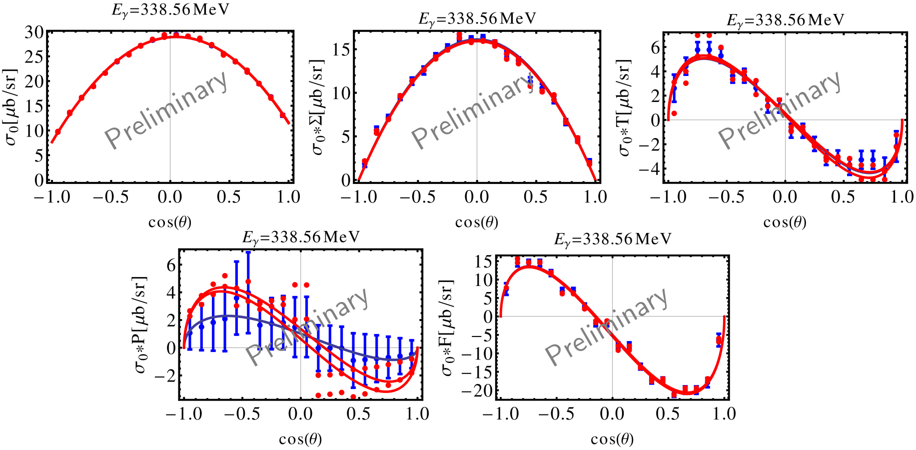
$$E_\gamma^{\mathrm{LAB}} \simeq E_\gamma^{\Delta} \simeq 338\ \mathrm{MeV}$$



Generate 1 more dataset.

TPWA fits using the bootstrapping method

$$E_\gamma^{\mathrm{LAB}} \simeq E_\gamma^{\Delta} \simeq 338 \text{ MeV}$$



Fit the additional dataset.

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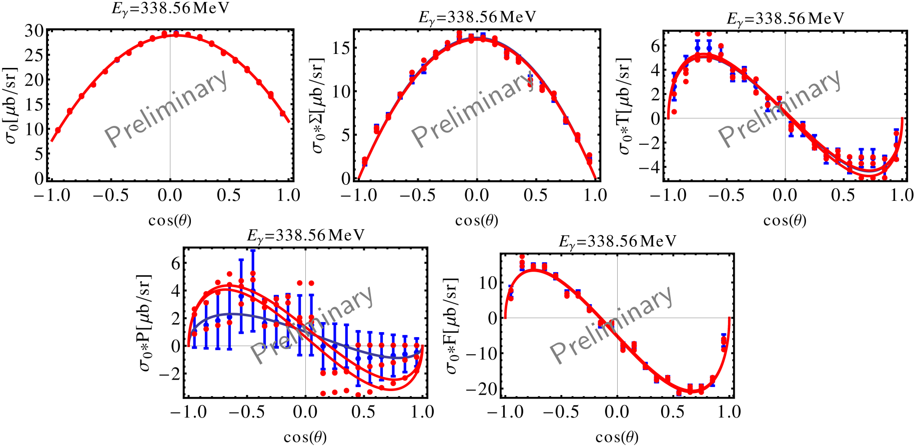

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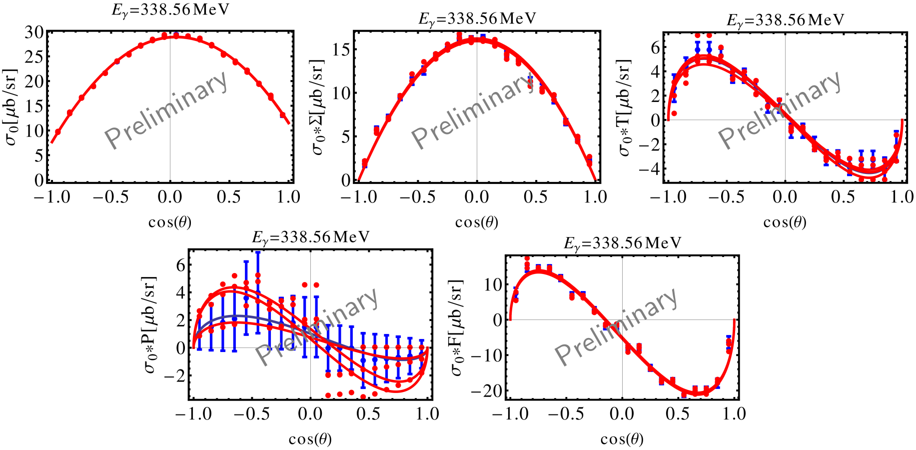

Fit the additional dataset.

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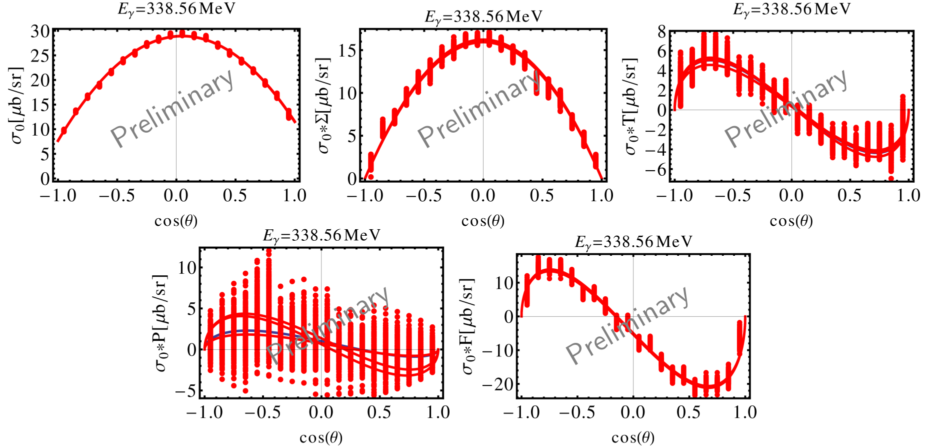

In total, 250 additional datasets are generated.

TPWA fits using the bootstrapping method

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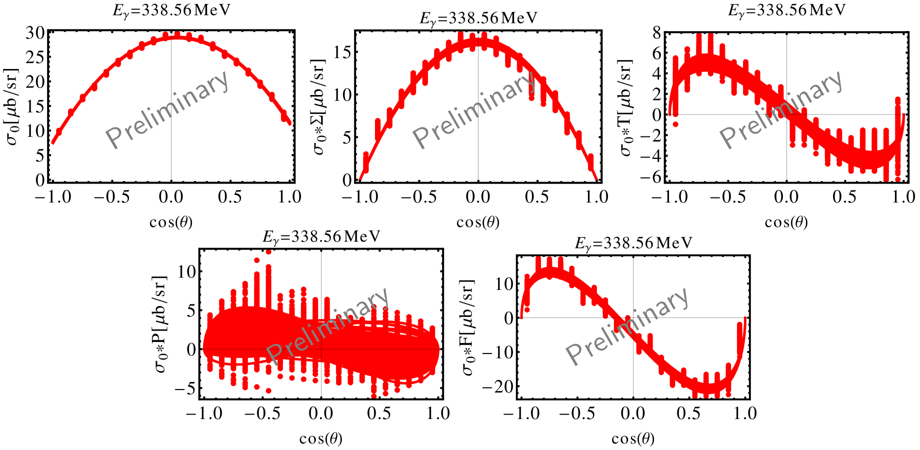
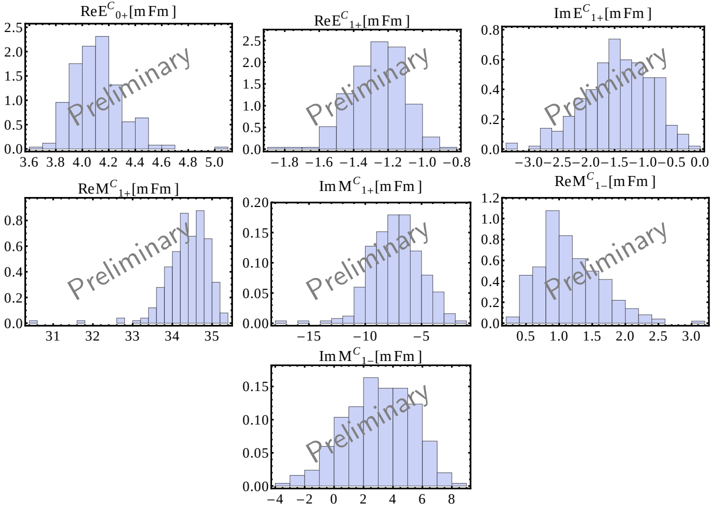


All of the $(1 + 250)$ datasets are fitted.

The TPWA fit step 2 is then applied to each one (for $\ell_{\mathrm{max}} = 1$).

Results for fits to data I

$\gamma p \rightarrow \pi^0 p$: $\underline{\{\sigma_0, \Sigma, T, F\}}$ from MAMI and $\underline{P}$ from World Data.
Histogram results for an Ensemble of $(1 + 250)$ datasets at $E_\gamma^{\mathrm{LAB}} \simeq 338\,\mathrm{MeV}$:



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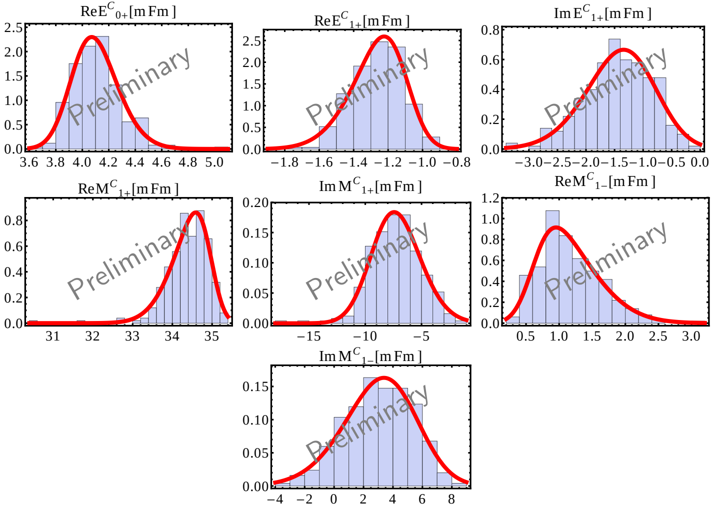

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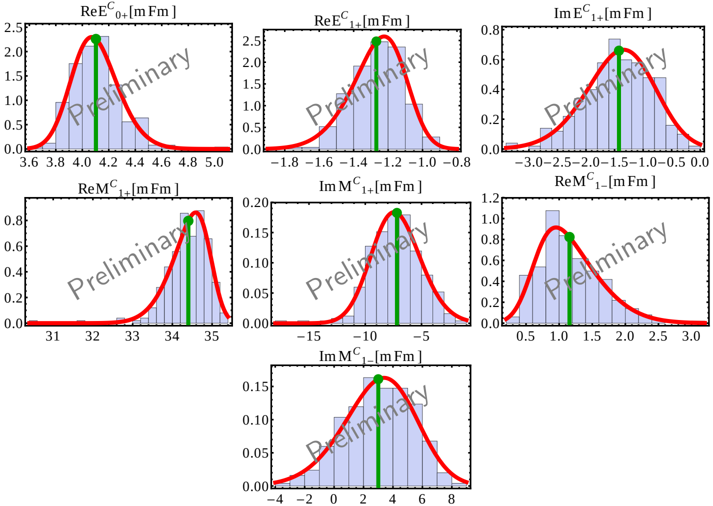

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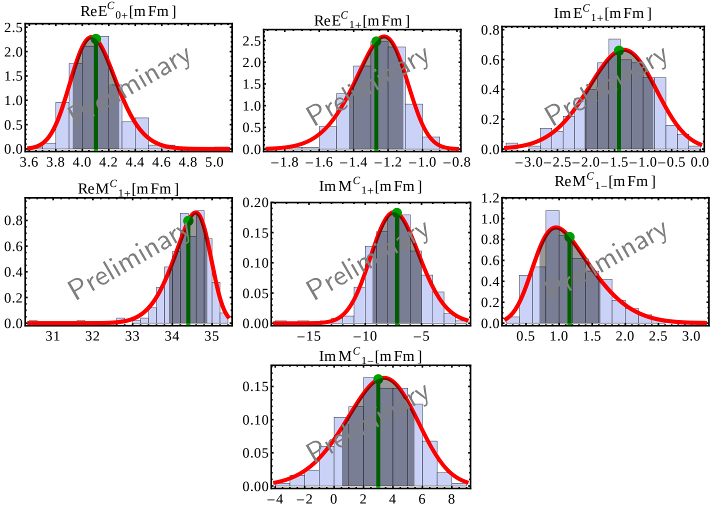

Results for fits to real data II

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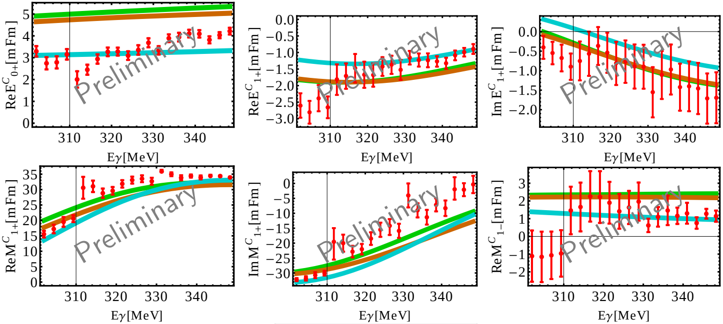
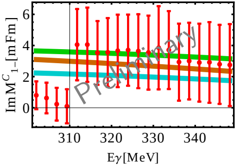


$E_\gamma^{\mathrm{LAB}} \in [300, 350]$ MeV



Green: MAID
Brown: SAID
Cyan: BnGa

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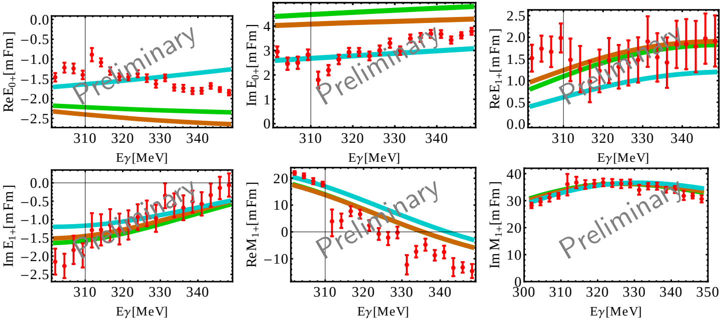
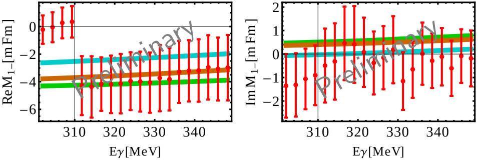


E_{0+} rotated to MAID phase



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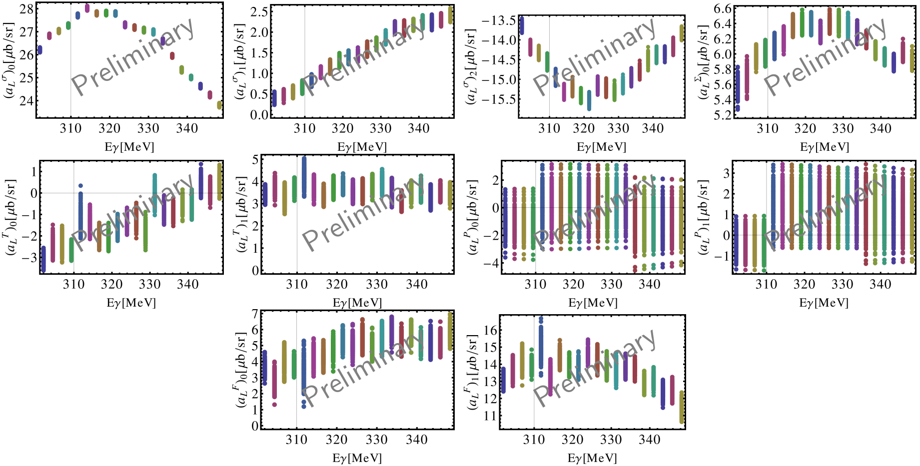
$\longrightarrow$ Investigations of higher ℓ_{max} as well as π^0 photoproduction data over the whole Δ-region are planned.
Important: D-waves have to be fitted or fixed to a model in order to obtain really correct S- and P-waves, due to interferences.

Thank You!

Appendix: Distributions of Legendre coefficients



Appendix: Does not knowing $\phi^{F}\left(W,\theta\right)$ cause harm?

I. For non. rel. QM / Spinless scattering:

$$f\left(W,\theta\right)=\sum_{\ell=0}^{\infty}\left(2\ell+1\right)f_{\ell}\left(W\right)P_{\ell}\left(\cos\theta\right)\leftrightarrow f_{\ell}\left(W\right)=\frac{1}{2}\int_{-1}^{1}d\cos\theta f(W,\theta)P_{\ell}\left(\cos\theta\right)$$

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II. There exist more involved projections for photoproduction, e.g.:

$$M_{\ell+}(W) = \frac{1}{2(\ell+1)}\int_{-1}^{1} d\cos\theta \Big[F_1(W,\theta) P_\ell(\cos\theta) - F_2(W,\theta) P_{\ell+1}(\cos\theta) - F_3(W,\theta) \frac{P_{\ell-1}(\cos\theta) - P_{\ell+1}(\cos\theta)}{2\ell+1} \Big]$$

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$$= \frac{1}{2(\ell+1)}\int_{-1}^{1} d\cos\theta \Big[\tilde{F}_1 e^{i\phi^F} P_\ell(\cos\theta) - \tilde{F}_2 e^{i\phi^F} P_{\ell+1}(\cos\theta) - \tilde{F}_3 e^{i\phi^F} \frac{P_{\ell-1}(\cos\theta) - P_{\ell+1}(\cos\theta)}{2\ell+1} \Big]$$

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$$f(W,\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell(W) P_\ell(\cos\theta) \leftrightarrow f_\ell(W) = \frac{1}{2}\int_{-1}^{1} d\cos\theta f(W,\theta) P_\ell(\cos\theta)$$

II. There exist more involved projections for photoproduction, e.g.:

$$\begin{aligned} M_{\ell+}(W) &= \frac{1}{2(\ell+1)}\int_{-1}^{1} d\cos\theta \Big[F_1(W,\theta) P_\ell(\cos\theta) - F_2(W,\theta) P_{\ell+1}(\cos\theta) \\ &\qquad - F_3(W,\theta)\frac{P_{\ell-1}(\cos\theta) - P_{\ell+1}(\cos\theta)}{2\ell+1}\Big] \\ &= \frac{1}{2(\ell+1)}\int_{-1}^{1} d\cos\theta \underbrace{e^{i\phi^F(W,\theta)}}_{\text{unknown}} \Big[\tilde{F}_1(W,\theta) P_\ell(\cos\theta) - \tilde{F}_2(W,\theta) P_{\ell+1}(\cos\theta) \\ &\qquad - \tilde{F}_3(W,\theta)\frac{P_{\ell-1}(\cos\theta) - P_{\ell+1}(\cos\theta)}{2\ell+1}\Big] \end{aligned}$$

$\longrightarrow$ Not knowing $\phi^F(W,\theta)$ denies access to partial waves via the full amplitudes!