Unique solutions of truncated partial wave analyses and complete experiments

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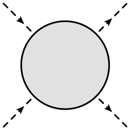
Introduction

Consider spin-less $2 \rightarrow 2$-amplitude: $A(W, \theta) =$ 

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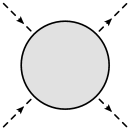
Consider spin-less $2 \to 2$-amplitude: $A(W, \theta) =$ 

$*)$ One single *observable* in scattering experiment: diff. cross section

$$\sigma_0(W, \theta) = |A(W, \theta)|^2.$$

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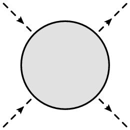
Consider spin-less $2 \to 2$-amplitude: $A(W, \theta) =$ 

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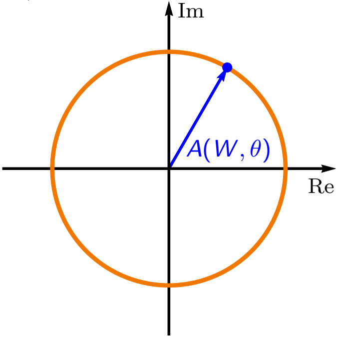
$\Rightarrow$ Complete experiment analysis: $|A(W, \theta)| = \sqrt{\sigma_0(W, \theta)}$, i.e. the amplitude lies on a circle.



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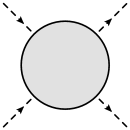
Consider spin-less $2 \to 2$-amplitude: $A(W, \theta) =$ 

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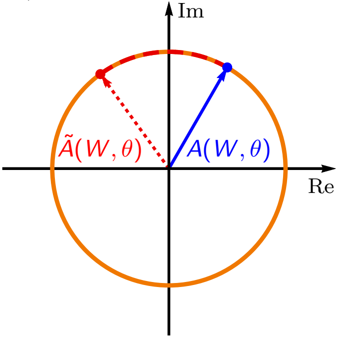

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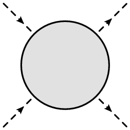
Consider spin-less $2 \rightarrow 2$-amplitude: $A(W, \theta) =$ 

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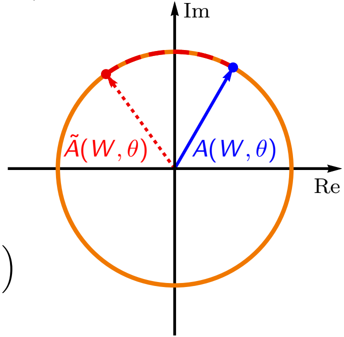
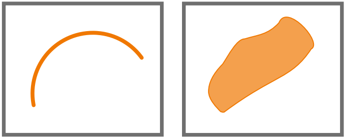
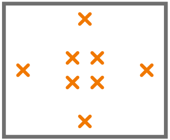


$\Rightarrow$ Implications for partial wave decomp.
$A(W, \theta) = \sum_{\ell=0}^{\infty} (2\ell + 1) A_\ell(W) P_\ell(\cos\theta)$,
$\left(\Leftrightarrow A_\ell(W) = \frac{1}{2} \int_{-1}^{1} d\cos\theta \, A(W, \theta) P_\ell(\cos\theta) \right)$
and in particular for truncated PWA?

Continuum- vs. discrete ambiguities

	Continuum ambiguities	Discrete ambiguities
∗) Definition:	$\boxed{\tilde{A}(W,\theta) = e^{i\Phi(W,\theta)} A(W,\theta)}$ [Bowcock & Burkhardt], [L. P. Kok], ...	For $A(W,\theta) = \hat{A}(W,\theta)\left(\cos\theta - \alpha\right)$, conjugate the zero/root: $\boxed{\alpha \rightarrow \alpha^{*}}$. [A. Gersten], [E. Barrelet], [L. P. Kok], [A. S. Omelaenko], ...
∗) Invariance:	$\sigma_0 = \lvert A \rvert^2 = A^{*} A$ $\rightarrow \tilde{A}^{*}\tilde{A} = e^{-i\Phi} A^{*} e^{i\Phi} A$ $= e^{i(\Phi - \Phi)} A^{*} A = A^{*} A = \sigma_0$ ✓	$\sigma_0 = \lvert \hat{A} \rvert^2 \left(\cos\theta - \alpha^{*}\right)\left(\cos\theta - \alpha\right)$ $\rightarrow \lvert \hat{A} \rvert^2 \left(\cos\theta - \left[\alpha^{*}\right]^{*}\right)\left(\cos\theta - \alpha^{*}\right)$ $= \lvert \hat{A} \rvert^2 \left(\cos\theta - \alpha^{*}\right)\left(\cos\theta - \alpha\right)$ $= \sigma_0$ ✓
∗) Illustration:		

Grey box: space of partial wave amplitudes $\{A_0, \ldots, A_\infty\}$, or $\{A_0, \ldots, A_L\}$.

Orange: parameter-regions of ambiguity, i.e. with same σ_0.

Continuum- vs. discrete ambiguities

Continuum ambiguities

$*)$ Definition:

$$\boxed{\tilde{A}(W,\theta) = e^{i\Phi(W,\theta)} A(W,\theta)}$$

[Bowcock & Burkhardt], [L. P. Kok], . . .

$*)$ Invariance:

$$\sigma_0 = |A|^2 = A^* A$$
$$\rightarrow \tilde{A}^* \tilde{A} = e^{-i\Phi} A^* e^{i\Phi} A$$
$$= e^{i(\Phi - \Phi)} A^* A = A^* A = \sigma_0 \checkmark$$

Discrete ambiguities

Definition: For $A(W,\theta) = \hat{A}(W,\theta)\left(\cos\theta - \alpha\right)$, conjugate the zero/root: $\boxed{\alpha \rightarrow \alpha^*}$.

[A. Gersten], [E. Barrelet], [L. P. Kok], [A. S. Omelaenko], . . .

Invariance:

$$\sigma_0 = \left|\hat{A}\right|^2 \left(\cos\theta - \alpha^*\right)\left(\cos\theta - \alpha\right)$$
$$\rightarrow \left|\hat{A}\right|^2 \left(\cos\theta - \left[\alpha^*\right]^*\right)\left(\cos\theta - \alpha^*\right)$$
$$= \left|\hat{A}\right|^2 \left(\cos\theta - \alpha^*\right)\left(\cos\theta - \alpha\right)$$
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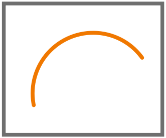
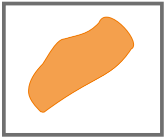
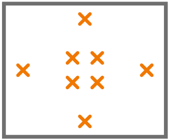
$*)$ Illustration:







Now: consider only mathematical ambiguities, disregarding physical constraints (e.g. unitarity!). Are discrete and continuum ambiguities different/related?

Effects of the full continuum ambiguity

$*)$ Transform $A(W,\theta) \longrightarrow \tilde{A}(W,\theta) := e^{i\Phi(W,\theta)} A(W,\theta)$ & write a Legendre-series for the rotation-function

$$e^{i\Phi(W,\theta)} = \sum_{k=0}^{\infty} L_k(W) P_k(\cos\theta).$$

How are the partial waves $\tilde{A}_\ell$ of $\tilde{A}(W,\theta) = \sum_{\ell=0}^{\infty} (2\ell+1)\tilde{A}_\ell(W)P_\ell(\cos\theta)$ expressed in terms of A_ℓ from $A(W,\theta) = \sum_{\ell=0}^{\infty} (2\ell+1)A_\ell(W)P_\ell(\cos\theta)$?

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$\hookrightarrow$ Mixing-formula: $\boxed{\tilde{A}_\ell(W) = \sum_{k=0}^{\infty} L_k(W) \sum_{m=|k-\ell|}^{k+\ell} \langle k,0;\ell,0|m,0\rangle^2 A_m(W)},$

$\langle j_1, m_1; j_2, m_2 | j, m\rangle$: Glebsch-Gordan coefficients.

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Explicitly: $$\tilde{A}_0(W) = \mathbf{L_0(W) A_0(W)} + L_1(W) A_1(W) + L_2(W) A_2(W) + \ldots,$$

$$\tilde{A}_1(W) = \mathbf{L_0(W) A_1(W)} + L_1(W) \left[\frac{1}{3} A_0(W) + \frac{2}{3} A_2(W) \right]$$
$$+ L_2(W) \left[\frac{2}{5} A_1(W) + \frac{3}{5} A_3(W) \right] + \ldots,$$

$$\tilde{A}_2(W) = \mathbf{L_0(W) A_2(W)} + L_1(W) \left[\frac{2}{5} A_1(W) + \frac{3}{5} A_3(W) \right]$$
$$+ L_2(W) \left[\frac{1}{5} A_0(W) + \frac{2}{7} A_2(W) + \frac{18}{35} A_4(W) \right] + \ldots.$$

$$\vdots$$

Effects of the full continuum ambiguity

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Explicitly:
$$\tilde{A}_0 = \mathbf{L_0 A_0} + L_1 A_1 + L_2 A_2 + \ldots,$$
$$\tilde{A}_1 = \mathbf{L_0 A_1} + L_1 \left[\frac{1}{3} A_0 + \frac{2}{3} A_2\right] + L_2 \left[\frac{2}{5} A_1 + \frac{3}{5} A_3\right] + \ldots,$$
$$\tilde{A}_2 = \mathbf{L_0 A_2} + L_1 \left[\frac{2}{5} A_1 + \frac{3}{5} A_3\right] + L_2 \left[\frac{1}{5} A_0 + \frac{2}{7} A_2 + \frac{18}{35} A_4\right] + \ldots.$$
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$*)$ For angle-independent phase $\Phi(W,\theta) = \Phi(W)$:
$e^{i\Phi(W,\theta)} = e^{i\Phi(W)} \equiv L_0(W)$ and $\tilde{A}_\ell(W) = L_0(W) A_\ell(W) = e^{i\Phi(W)} A_\ell(W)$.
$\longrightarrow A_\ell(W)$ do not mix any more & are rotated by the same phase!

$*)$ Non-linearity introduced by the exp-function in the rotation $e^{i\Phi(W,\theta)}$ generates complicated mixings, even when the phase $\Phi(W,\theta)$ itself is *simple*, e.g. $\Phi(W,\theta) = a(W) + b(W)\cos\theta$.

Effects of the full continuum ambiguity

Illustration using a toy model: [arXiv:1706.03211v1]

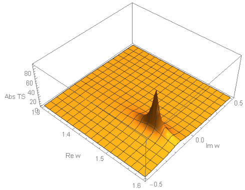
$$A(W,\theta) = T_S(W) + T_P(W)\cos(\theta),$$
$$T_{S,P}(W) = \frac{a_{S,P}}{M_{S,P} - i\Gamma_{S,P}/2 - W},$$

where

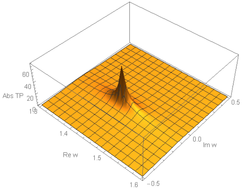
$$a_S = 0.5 + 0.4i;\, M_S = 1.535;\, \Gamma_S = 0.15,$$
$$a_P = 0.4 + 0.3i;\, M_P = 1.44;\, \Gamma_P = 0.1.$$

NON-ROTATED PW

S-wave



P-wave



D-wave

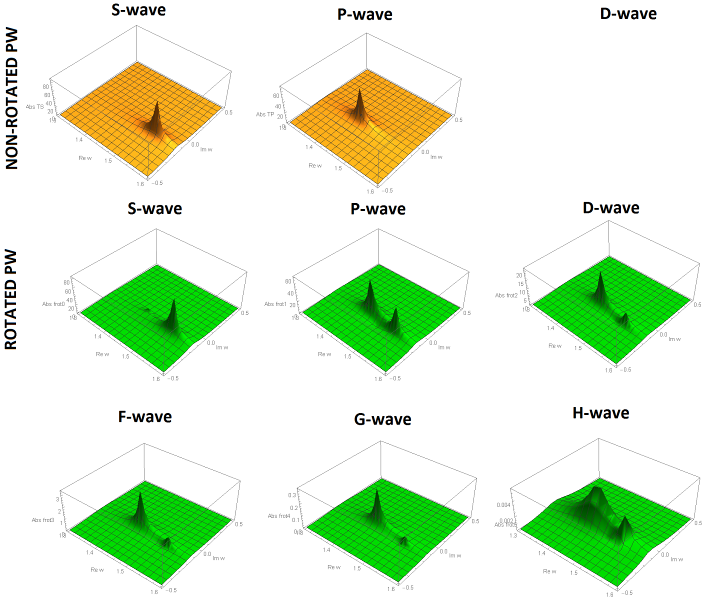
$\hookrightarrow$ Multiply this amplitude by a *simple* phase, e.g. $\exp\left[2. + 0.5\cos\theta\right]$.

Effects of the full continuum ambiguity

$$A(W,\theta)$$

$$\downarrow$$

$$e^{i(2.+0.5\cos\theta)} * A(W,\theta)$$



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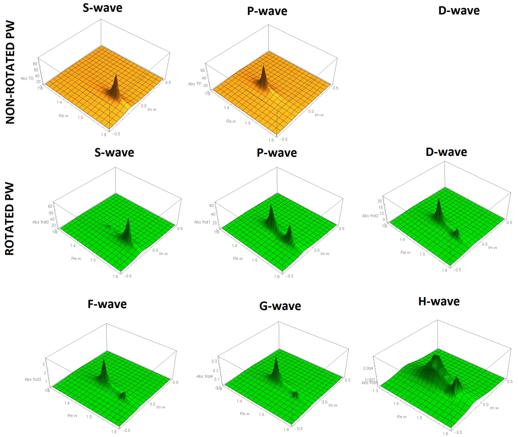

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Discrete ambiguities in scalar TPWAs

$*)$ A general truncated (i.e. polynomial-) amplitude for arbitrary L, $A = \sum_{\ell=0}^{L} (2\ell + 1) A_{\ell} P_{\ell}(\cos\theta)$, has the linear-factorization:

$$A = \lambda \left(\cos\theta - \alpha_1\right) \left(\cos\theta - \alpha_2\right) \ldots \left(\cos\theta - \alpha_L\right) \text{ , with } \lambda \propto A_L .$$

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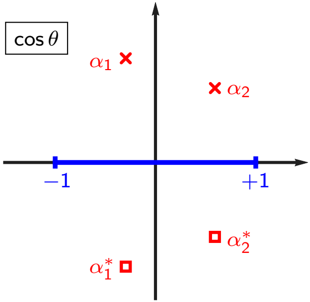
$*)$ roots $(\lambda, \{\alpha_i\}) \leftrightarrow$ partial waves $\{A_\ell\}$

$*)$ Define 'mappings' $\boldsymbol{\pi}_n$, which comprise all possibilities to complex conjugate subsets of the roots:

$$\alpha_i \longrightarrow \boldsymbol{\pi}_n \left(\alpha_i\right).$$

(Example for $L = 2$ on the right $\rightarrow$)

$\hookrightarrow$ There exist in total 2^L possibilities and thus maps $\boldsymbol{\pi}_n$.



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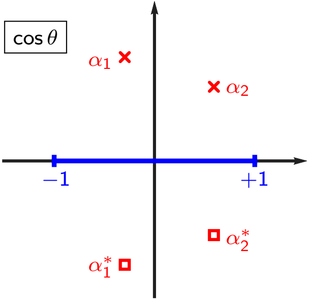

*) One can transform to 2^L ambiguous amplitudes:

$$A^{(n)} = \lambda \prod_{i=1}^{L} \left(\cos\theta - \boldsymbol{\pi}_n\left[\alpha_i\right]\right) \equiv \sum_{\ell=0}^{L} (2\ell+1) A_\ell^{(n)}(W) P_\ell(\cos\theta),$$

which <u>all</u> have the same c.s. $\sigma_0 = \left|\lambda\right|^2 \prod_{i=1}^{L} \left(\cos\theta - \alpha_i^*\right)\left(\cos\theta - \alpha_i\right)$.

Continuum- vs. discrete ambiguities in scalar TPWAs

Are these discrete ambiguities in any way connected to the continuum ambiguities discussed before?

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⇒ Yes: discrete ambiguities *are* angle-dependent rotations, for a certain discrete set of 2^L phase-rotations $\Phi_n(W,\theta)$:

$$\boxed{e^{i\Phi_n(W,\theta)} = \frac{A^{(n)}(W,\theta)}{A(W,\theta)} = \frac{(\cos\theta - \boldsymbol{\pi}_n[\alpha_1])\ldots(\cos\theta - \boldsymbol{\pi}_n[\alpha_L])}{(\cos\theta - \alpha_1)\ldots(\cos\theta - \alpha_L)}}$$

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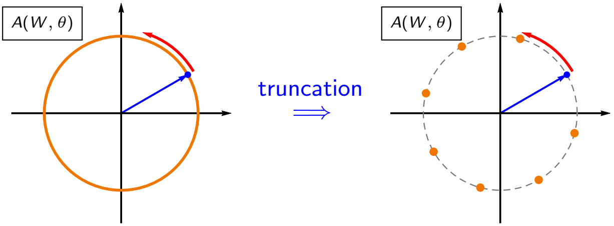
$*)$ Illustration: discrete ambiguities are a *remnant* of the continuum ambiguity



$\hookrightarrow$ Therefore, discrete ambiguities mix partial waves as well!

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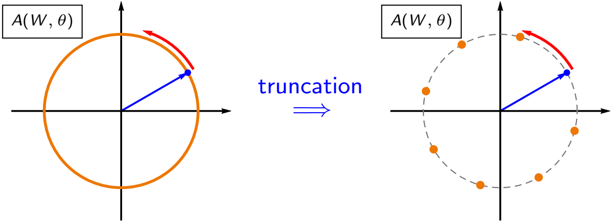

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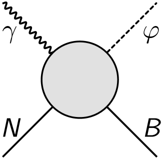
Now: Look at a reaction involving particles with spin!

Photoproduction amplitudes

Photoproduction amplitude in the CMS:



$$= \mathcal{T}_{fi} = \mathcal{C}\chi^{\dagger}_{m_{s_f}} \Big[i\vec{\sigma}\cdot\hat{\epsilon}F_1 + \vec{\sigma}\cdot\hat{q}\vec{\sigma}\cdot\left(\hat{k}\times\hat{\epsilon}\right)F_2 + i\vec{\sigma}\cdot\hat{k}\hat{q}\cdot\hat{\epsilon}F_3 + i\vec{\sigma}\cdot\hat{q}\hat{q}\cdot\hat{\epsilon}F_4 \Big] \chi_{m_{s_i}}$$

[Chew, Goldberger, Low & Nambu (1957)]

$\rightarrow$ Process fully described by 4 complex amplitudes $F_i\left(W,\theta\right)$.

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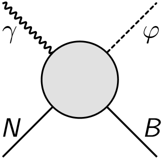

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[Chew, Goldberger, Low & Nambu (1957)]

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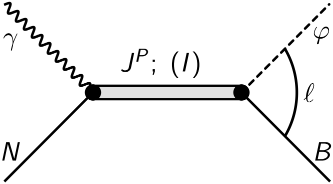
Important concept: expansion of full amplitudes into partial waves:

$$F_1\left(W,\theta\right) = \sum_{\ell=0}^{\infty}\left\{ \left[\ell M_{\ell+} + E_{\ell+}\right]P^{'}_{\ell+1}\left(\cos\left(\theta\right)\right) + \left[\left(\ell+1\right)M_{\ell-} + E_{\ell-}\right]P^{'}_{\ell-1}\left(\cos\left(\theta\right)\right) \right\}$$

$$F_2\left(W,\theta\right) = \ldots$$



$*)$ $J = |\ell \pm 1/2|$, $P = (-)^{\ell+1}$.

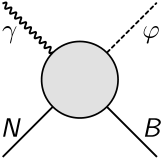
$*)$ s-chn. resonance $J^P;\ (I)$

$\updownarrow$

multipole $E^{(I)}_{\ell\pm},\ M^{(I)}_{\ell\pm}$

Photoproduction amplitudes

Photoproduction amplitude in the CMS:



$$= \mathcal{T}_{fi} = \mathcal{C}\chi^{\dagger}_{m_{s_f}} \Big[i\vec{\sigma}\cdot\hat{\epsilon}F_1 + \vec{\sigma}\cdot\hat{q}\vec{\sigma}\cdot\left(\hat{k}\times\hat{\epsilon}\right)F_2 + i\vec{\sigma}\cdot\hat{k}\hat{q}\cdot\hat{\epsilon}F_3 + i\vec{\sigma}\cdot\hat{q}\hat{q}\cdot\hat{\epsilon}F_4 \Big]\chi_{m_{s_i}}$$

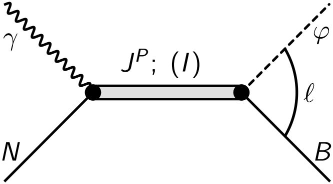
[Chew, Goldberger, Low & Nambu (1957)]

$\rightarrow$ Process fully described by 4 complex amplitudes $F_i\left(W,\theta\right)$.

Important concept: expansion of full amplitudes into partial waves:

$$F_1\left(W,\theta\right) = \sum_{\ell=0}^{\cancel{\infty}\,L} \left\{ \left[\ell M_{\ell+} + E_{\ell+}\right] P^{'}_{\ell+1}\left(\cos\left(\theta\right)\right) + \left[\left(\ell+1\right)M_{\ell-} + E_{\ell-}\right] P^{'}_{\ell-1}\left(\cos\left(\theta\right)\right) \right\}$$

$$F_2\left(W,\theta\right) = \ldots$$



<u>In practice:</u>

Truncate at some finite L

$\rightarrow$ Try to extract the $4L$ complex multipoles in a fit to the data.

Polarization observables

Generic definition of an observable

$$\Omega = \frac{\beta}{\sigma_0} \left[\left(\frac{d\sigma}{d\Omega} \right)^{(B_1, T_1, R_1)} - \left(\frac{d\sigma}{d\Omega} \right)^{(B_2, T_2, R_2)} \right]$$

$*$) In total, 16 non-redundant observables

$$\Omega^{\alpha} \left(W, \theta \right) = \frac{1}{2\sigma_0} \sum_{i,j} F_i^{*} \hat{A}_{ij}^{\alpha} F_j, \quad \alpha = 1, \ldots, 16$$

can be defined, involving Beam-, Target- and Recoil Polarization.

Beam		Target			Recoil			Target + Recoil			
	-	-	-	-	x'	y'	z'	x'	x'	z'	z'
	-	x	y	z	-	-	-	x	z	x	z
unpolarized	$\left(\frac{d\sigma}{d\Omega} \right)_0 = \sigma_0$		T			P		$T_{x'}$	$L_{x'}$	$T_{z'}$	$L_{z'}$
linear	Σ	H	P	G	$O_{x'}$	T	$O_{z'}$				
circular		F		E	$C_{x'}$		$C_{z'}$				

Observables in the transversity basis

Observable	Transversity representation	Type
σ_0	$\frac{1}{2}\left(\lvert b_1\rvert^2 + \lvert b_2\rvert^2 + \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$	
$\check{\Sigma}$	$\frac{1}{2}\left(-\lvert b_1\rvert^2 - \lvert b_2\rvert^2 + \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$	$\mathcal{S}$
$\check{T}$	$\frac{1}{2}\left(\lvert b_1\rvert^2 - \lvert b_2\rvert^2 - \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$	
$\check{P}$	$\frac{1}{2}\left(-\lvert b_1\rvert^2 + \lvert b_2\rvert^2 - \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$	
$\check{G}$	$\mathrm{Im}\left[-b_1 b_3^* - b_2 b_4^*\right]$	
$\check{H}$	$-\mathrm{Re}\left[b_1 b_3^* - b_2 b_4^*\right]$	$\mathcal{BT}$
$\check{E}$	$-\mathrm{Re}\left[b_1 b_3^* + b_2 b_4^*\right]$	
$\check{F}$	$\mathrm{Im}\left[b_1 b_3^* - b_2 b_4^*\right]$	
$\check{O}_{x'}$	$-\mathrm{Re}\left[-b_1 b_4^* + b_2 b_3^*\right]$	
$\check{O}_{z'}$	$\mathrm{Im}\left[-b_1 b_4^* - b_2 b_3^*\right]$	$\mathcal{BR}$
$\check{C}_{x'}$	$\mathrm{Im}\left[b_1 b_4^* - b_2 b_3^*\right]$	
$\check{C}_{z'}$	$\mathrm{Re}\left[b_1 b_4^* + b_2 b_3^*\right]$	
$\check{T}_{x'}$	$-\mathrm{Re}\left[-b_1 b_2^* + b_3 b_4^*\right]$	
$\check{T}_{z'}$	$-\mathrm{Im}\left[b_1 b_2^* - b_3 b_4^*\right]$	$\mathcal{TR}$
$\check{L}_{x'}$	$-\mathrm{Im}\left[-b_1 b_2^* - b_3 b_4^*\right]$	
$\check{L}_{z'}$	$\mathrm{Re}\left[-b_1 b_2^* - b_3 b_4^*\right]$	

$*)$ Transversity amplitudes: $b_i = \sum_j M_{ij} F_j$.

$*)$ Different scheme of spin-quantization:

$$\left\langle m_{s_f} \right| \mathcal{T} \left| m_{s_i} \right\rangle \updownarrow \left\langle t_f \right| \mathcal{T} \left| t_i \right\rangle .$$

$t_i(t_f) = \pm\frac{1}{2}$: spin-projection of initial (final) baryon on the normal of the reaction plane.

$*)$ Observables simplify:

$$\check{\Omega}^{\alpha} = \frac{1}{2} \sum_{i,j} b_i^* \tilde{\Gamma}^{\alpha}_{ij} b_j .$$

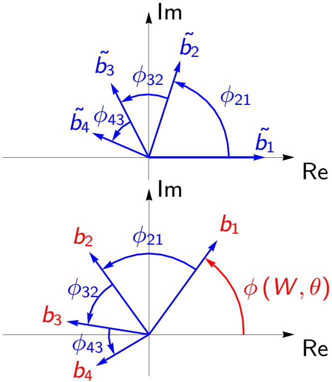
Complete experiments

$*$) <u>Question:</u> How many and which observables $\check{\Omega}^{\alpha}$ have to be measured in order to uniquely extract the full amplitudes (e.g. transversity amplitudes b_i)?.

Complete experiments

$\ast$) Mathematical solution: [Chiang & Tabakin, Phys. Rev. C 55, 2054 (1997)]
Utilize b.t.p.-form $\check{\Omega}^{\alpha} = \frac{1}{2} \sum_{i,j} b_i^* \tilde{\Gamma}^{\alpha}_{ij} b_j$ and the completeness of the $\tilde{\Gamma}^{\alpha}$-matrices ($\tilde{\Gamma}^{\alpha}$ form an orthonormal basis): $\frac{1}{4} \sum_{\alpha} \tilde{\Gamma}^{\alpha}_{ba} \tilde{\Gamma}^{\alpha}_{st} = \delta_{as} \delta_{bt}$

$$b_i^* b_j = \frac{1}{2} \sum_{\alpha} \left(\tilde{\Gamma}^{\alpha}_{ij} \right)^* \check{\Omega}^{\alpha} \rightarrow |b_i| = \sqrt{b_i^* b_i} \;\; \& \;\; e^{\phi_{ij}} = \frac{b_j^* b_i}{|b_j| \, |b_i|}$$



Complete experiments

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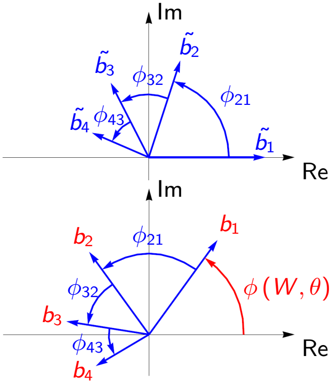
$$b_i^{*} b_j = \frac{1}{2} \sum_{\alpha} \left(\tilde{\Gamma}^{\alpha}_{ij} \right)^{*} \check{\Omega}^{\alpha} \rightarrow \left| b_i \right| = \sqrt{b_i^{*} b_i} \;\; \& \;\; e^{\phi_{ij}} = \frac{b_j^{*} b_i}{\left| b_j \right| \left| b_i \right|}$$

*) Use "Fierz-identities" $\check{\Omega}^{\alpha} \check{\Omega}^{\beta} = \mathcal{C}^{\alpha\beta}_{\delta\eta} \check{\Omega}^{\delta} \check{\Omega}^{\eta}$ (with known coefficients $\mathcal{C}^{\alpha\beta}_{\delta\eta}$) to prove:

- 8 observables can yield $|b_i|$ & ϕ_{ij}.
- Double-polarization obs. with recoil-polarization (type $\mathcal{BR}$ and $\mathcal{TR}$) have to be measured.
- No more than two observables from the same double-polarization class are allowed.
- The phase $\phi(W, \theta)$ remains undetermined.



Complete experiments

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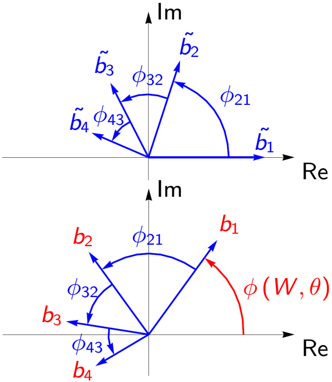
$\ast$) Use "Fierz-identities" $\check{\Omega}^{\alpha} \check{\Omega}^{\beta} = \mathcal{C}^{\alpha\beta}_{\delta\eta} \check{\Omega}^{\delta} \check{\Omega}^{\eta}$ (with known coefficients $\mathcal{C}^{\alpha\beta}_{\delta\eta}$) to prove:

- <u>8 observables</u> can yield $\left| b_i \right|$ & ϕ_{ij}.

$\hookrightarrow$ Ask a similar question for the TPWA: i.e., how many and which observables can uniquely fix the *multipoles* $\left\{ E_{\ell\pm}, M_{\ell\pm} \right\}$?



Complete experiments in a TPWA I

$\ast)$ Consider the group $\mathcal{S}$ observables:

$$\sigma_0 = \frac{1}{2}\left(\left|b_1\right|^2 + \left|b_2\right|^2 + \left|b_3\right|^2 + \left|b_4\right|^2 \right),\ \check{\Sigma} = \frac{1}{2}\left(- \left|b_1\right|^2 - \left|b_2\right|^2 + \left|b_3\right|^2 + \left|b_4\right|^2 \right)$$

$$\check{T} = \frac{1}{2}\left(\left|b_1\right|^2 - \left|b_2\right|^2 - \left|b_3\right|^2 + \left|b_4\right|^2 \right),\ \check{P} = \frac{1}{2}\left(- \left|b_1\right|^2 + \left|b_2\right|^2 - \left|b_3\right|^2 + \left|b_4\right|^2 \right)$$

Complete experiments in a TPWA I

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$\ast$) These 4 observables are invariant under <u>4-fold</u> continuum ambiguities:

$b_j(W,\theta) \longrightarrow e^{i\Phi_j(W,\theta)} b_j(W,\theta)$, with *different* phases Φ_j, $j = 1, \ldots, 4$.

Complete experiments in a TPWA I

*) Consider the group $\mathcal{S}$ observables:

$$\sigma_0 = \frac{1}{2}\left(\left|b_1\right|^2 + \left|b_2\right|^2 + \left|b_3\right|^2 + \left|b_4\right|^2\right), \ \check{\Sigma} = \frac{1}{2}\left(-\left|b_1\right|^2 - \left|b_2\right|^2 + \left|b_3\right|^2 + \left|b_4\right|^2\right)$$

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*) Linear factorizations in a TPWA truncated at L for the b_i (here: $t = \tan\frac{\theta}{2}$):

$$b_1(\theta) = -\mathcal{C}\, a_{2L} \frac{\exp\left(-i\frac{\theta}{2}\right)}{\left(1+t^2\right)^L} \prod_{j=1}^{2L}\left(t+\beta_j\right), \ b_2(\theta) = -\mathcal{C}\, a_{2L} \frac{\exp\left(i\frac{\theta}{2}\right)}{\left(1+t^2\right)^L} \prod_{j=1}^{2L}\left(t-\beta_j\right),$$

$$b_3(\theta) = \mathcal{C}\, a_{2L} \frac{\exp\left(-i\frac{\theta}{2}\right)}{\left(1+t^2\right)^L} \prod_{k=1}^{2L}\left(t+\alpha_k\right), \ b_4(\theta) = \mathcal{C}\, a_{2L} \frac{\exp\left(i\frac{\theta}{2}\right)}{\left(1+t^2\right)^L} \prod_{k=1}^{2L}\left(t-\alpha_k\right).$$

We have: roots $\{\alpha_k, \beta_j\} \leftrightarrow$ multipoles $\{E_{\ell\pm}, M_{\ell\pm}\}$.

$\hookrightarrow$ Can we mimic the same (root-) conjugation procedure as in the scalar case?

Complete experiments in a TPWA II

Yes: We now have 4^{2L} 'mappings' $\boldsymbol{\pi}_n$ that parametrize all possible conjugations:

$$\alpha_k \longrightarrow \boldsymbol{\pi}_n(\alpha_k),\ \beta_j \longrightarrow \boldsymbol{\pi}_n(\beta_j).$$

Complete experiments in a TPWA II

Yes: We now have 4^{2L} 'mappings' $\boldsymbol{\pi}_n$ that parametrize all possible conjugations:

$$\alpha_k \longrightarrow \boldsymbol{\pi}_n(\alpha_k),\ \beta_j \longrightarrow \boldsymbol{\pi}_n(\beta_j).$$

$*)$ These discrete ambiguities are generated by the following phase-rotations:

$$e^{i\Phi_1(W,\theta)} = \prod_{j=1}^{2L} \frac{(t + \boldsymbol{\pi}_n[\beta_j])}{(t + \beta_j)},\ e^{i\Phi_2(W,\theta)} = \prod_{j=1}^{2L} \frac{(t - \boldsymbol{\pi}_n[\beta_j])}{(t - \beta_j)},$$

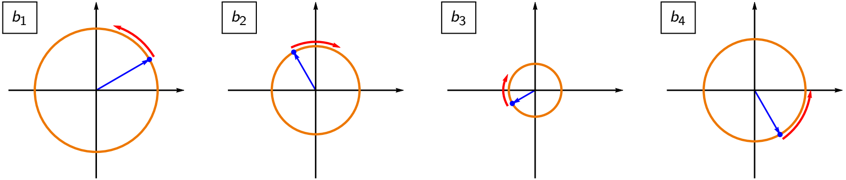
$$e^{i\Phi_3(W,\theta)} = \prod_{k=1}^{2L} \frac{(t + \boldsymbol{\pi}_n[\alpha_k])}{(t + \alpha_k)},\ e^{i\Phi_4(W,\theta)} = \prod_{k=1}^{2L} \frac{(t - \boldsymbol{\pi}_n[\alpha_k])}{(t - \alpha_k)}.$$

These rotations are *explicitly* of 4-fold type:

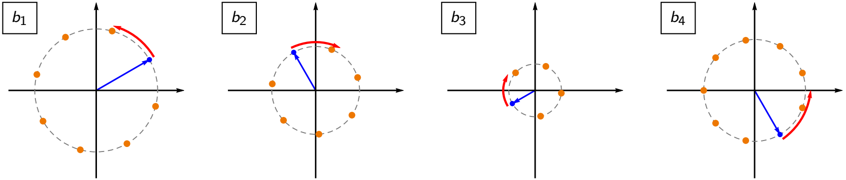
$$b_j(W,\theta) \longrightarrow e^{i\Phi_j(W,\theta)} b_j(W,\theta),\ j = 1,\ldots,4.$$

Complete experiments in a TPWA II

Illustration:

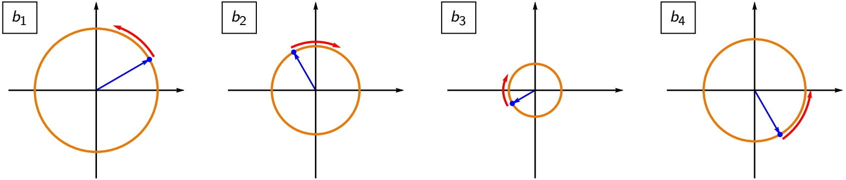


$\Downarrow$ truncation



Complete experiments in a TPWA II

Illustration:



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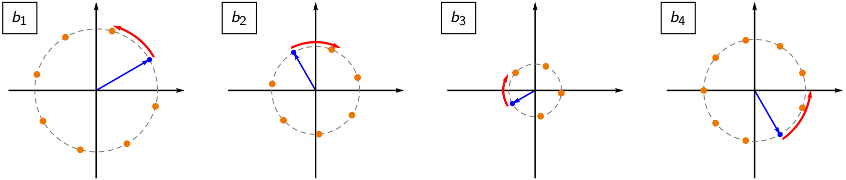

$\hookrightarrow$ The relative-phases ϕ^b_{ij} of the b_i will <u>change</u> under these ambiguities!

$\ast$) Double-polarization observables can help with that problem!

Complete experiments in a TPWA III

$*$) The group $\mathcal{S}$ observables $\left\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\right\}$ have discrete ambiguities in a TPWA that correspond to 4−fold phase-rotations acting on the $b_i(W, \theta)$.

$\hookrightarrow$ Analyze additional observables, which are sensitive to the relative phases ϕ_{ij}^{b} affected by the ambiguities. Try for instance the $\mathcal{BT}$-observables:

$$\check{E} = -\mathrm{Re}\left[b_1 b_3^* + b_2 b_4^*\right], \quad \check{H} = -\mathrm{Re}\left[b_1 b_3^* - b_2 b_4^*\right],$$

$$\check{G} = \mathrm{Im}\left[-b_1 b_3^* - b_2 b_4^*\right], \quad \check{F} = \mathrm{Im}\left[b_1 b_3^* - b_2 b_4^*\right].$$

Beam		Target			Recoil			Target + Recoil			
	-	-	-	-	x'	y'	z'	x'	x'	z'	z'
	-	x	y	z	-	-	-	x	z	x	z
unpolarized	σ_0		T			P		$T_{x'}$	$L_{x'}$	$T_{z'}$	$L_{z'}$
linear	Σ	H	P	G	$O_{x'}$	T	$O_{z'}$			Incomplete	
circular		F		E	$C_{x'}$		$C_{z'}$				

Complete experiments in a TPWA III

$*)$ The group $\mathcal{S}$ observables $\left\{ \sigma_0, \check{\Sigma}, \check{T}, \check{P} \right\}$ have discrete ambiguities in a TPWA that correspond to $\underline{4-\text{fold}}$ phase-rotations acting on the $b_i(W, \theta)$.

$\hookrightarrow$ Analyze additional observables, which are sensitive to the relative phases ϕ_{ij}^{b} affected by the ambiguities. Try for instance the $\mathcal{BT}$-observables:

$$\check{E} = -\mathrm{Re}\left[b_1 b_3^* + b_2 b_4^* \right], \quad \check{H} = -\mathrm{Re}\left[b_1 b_3^* - b_2 b_4^* \right],$$
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Beam		Target			Recoil			Target + Recoil			
	-	-	-	-	x'	y'	z'	x'	x'	z'	z'
	-	x	y	z	-	-	-	x	z	x	z
unpolarized	σ_0		T			P		$T_{x'}$	$L_{x'}$	$T_{z'}$	$L_{z'}$
linear	Σ	H	P	G	$O_{x'}$	T	$O_{z'}$				
circular		F		E	$C_{x'}$		$C_{z'}$				

Complete

Complete experiments in a TPWA III

$\hookrightarrow$ Analyze additional observables, which are sensitive to the relative phases ϕ_{ij}^{b} affected by the ambiguities. Try for instance the $\mathcal{BT}$-observables:

$$\check{E} = -\mathrm{Re}\left[b_1 b_3^* + b_2 b_4^*\right], \quad \check{H} = -\mathrm{Re}\left[b_1 b_3^* - b_2 b_4^*\right],$$

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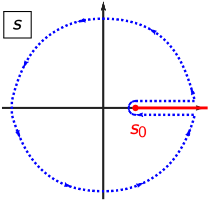
Beam		Target			Recoil			Target + Recoil			
	-	-	-	-	x'	y'	z'	x'	x'	z'	z'
	-	x	y	z	-	-	-	x	z	x	z
unpolarized	σ_0		T			P		$T_{x'}$	$L_{x'}$	$T_{z'}$	$L_{z'}$
linear	Σ	H	P	G	$O_{x'}$	T	$O_{z'}$			Complete	
circular		F		E	$C_{x'}$		$C_{z'}$				

(i) 'Complete sets of 5': understood algebraically and checked numerically.

(ii) 'Complete sets of <u>4</u>': found numerically, 'by accident' and <u>not</u> understood algebraically. [R. Workman, L. Tiator, Y.W., M. Döring, H. Haberzettl (2017)]

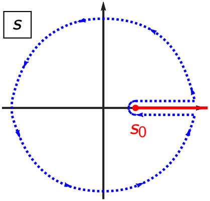
Resolving phase-ambiguities: analyticity



$*)$ Consider amplitude $A(s,t)$ with pertinent form of analyticity-constraint, i.e. *dispersion relation* in s:

$$\mathrm{Re}\left[A(s)\right] = \frac{1}{\pi}\,\hat{\mathbb{P}}\int_{s_0}^{\infty} ds' \frac{\mathrm{Im}\left[A\left(s'\right)\right]}{s'-s}.$$

Resolving phase-ambiguities: analyticity



∗) Consider amplitude $A(s,t)$ with pertinent form of analyticity-constraint, i.e. *dispersion relation* in s:

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∗) Assume the original amplitude $A(s)$ fulfills this constraint. Does there exist a phase-rotation $e^{i\phi(s)}$, such that $\tilde{A}(s) := e^{i\phi(s)}A(s)$ respects the <u>same</u> analyticity-constraint?

Resolving phase-ambiguities: analyticity

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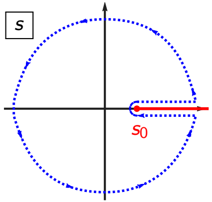
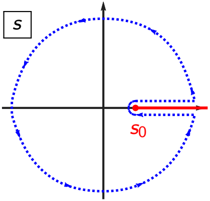


*) Assume the original amplitude $A(s)$ fulfills this constraint. Does there exist a phase-rotation $e^{i\phi(s)}$, such that $\tilde{A}(s) := e^{i\phi(s)}A(s)$ respects the <u>same</u> analyticity-constraint? $\rightarrow$ Integral-equation:

$$\sin\phi\left(s\right)\,\mathrm{Im}A\left(s\right) = \frac{1}{\pi}\,\hat{\mathbb{P}}\int_{s_0}^{\infty} ds'\,\frac{\left[\cos\phi\left(s\right)-\cos\phi\left(s'\right)\right]\,\mathrm{Im}A\left(s'\right)}{s'-s}$$
$$-\frac{1}{\pi^2}\,\hat{\mathbb{P}}\int_{s_0}^{\infty} ds'\,\hat{\mathbb{P}}\int_{s_0}^{\infty} d\tilde{s}\,\frac{\sin\phi\left(s'\right)\,\mathrm{Im}A\left(\tilde{s}\right)}{\left(s'-s\right)\left(\tilde{s}-s'\right)}.$$

$\hookrightarrow$ Does this equation have solutions for $e^{i\phi(s)}$ or $\phi(s)$? If yes, how many?

Resolving phase-ambiguities: analyticity



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$$\mathrm{Re}\left[A(s)\right] = \frac{1}{\pi}\,\hat{\mathbb{P}}\int_{s_0}^{\infty} ds'\,\frac{\mathrm{Im}\left[A\left(s'\right)\right]}{s'-s}.$$

$*)$ Assume the original amplitude $A(s)$ fulfills this constraint. Does there exist a phase-rotation $e^{i\phi(s)}$, such that $\tilde{A}(s) := e^{i\phi(s)}A(s)$ respects the <u>same</u> analyticity-constraint? $\rightarrow$ Integral-equation:

$$\begin{aligned}\sin\phi\left(s\right)\,\mathrm{Im}A\left(s\right) = & \frac{1}{\pi}\,\hat{\mathbb{P}}\int_{s_0}^{\infty} ds'\,\frac{\left[\cos\phi\left(s\right)-\cos\phi\left(s'\right)\right]\mathrm{Im}A\left(s'\right)}{s'-s} \\ & -\frac{1}{\pi^2}\,\hat{\mathbb{P}}\int_{s_0}^{\infty} ds'\,\hat{\mathbb{P}}\int_{s_0}^{\infty} d\tilde{s}\,\frac{\sin\phi\left(s'\right)\,\mathrm{Im}A\left(\tilde{s}\right)}{\left(s'-s\right)\left(\tilde{s}-s'\right)}.\end{aligned}$$

$\hookrightarrow$ Does this equation have solutions for $e^{i\phi(s)}$ or $\phi(s)$? If yes, how many?

$*)$ Formal treatment of amplitude-reconstruction using analyticity in <u>two</u> variables (s,t):

[I. Sabba Stefanescu, J. Math. Phys. **25** (6), 2052 (1984).] (tough paper!!!)

Challenge: Stefanescu-paper

On the construction of amplitudes with Mandelstam analyticity from observable quantities

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It is shown that the problem of the construction of scattering amplitudes with Mandelstam analyticity from knowledge of their modulus in the three physical channels can be reduced, within a rather large class of functions, to the second Cousin problem of the theory of functions of two complex variables. As a consequence, it can be solved completely and explicitly. We derive conditions on the modulus function, under which at least one solution exists, as well as criteria for the correct resolution of the discrete ambiguity at fixed energy.

PACS numbers: 11.50.Nk, 11.80.Gw, 11.20.Fm, 03.80. + r

I. INTRODUCTION

The problem of the determination of the phase of the scattering amplitude from observable quantities (i.e., $d\sigma/d\Omega$ for scattering of spinless particles, $d\sigma/d\Omega$ and polarization for spin-0-spin-$\frac{1}{2}$ scattering, etc.) has an obvious physical interest and has led, in the course of time, to a set of very elegant studies in mathematical physics.[1–8] These studies (see Ref. 9 for a review) have succeeded in establishing with precision the extent of the ambiguity that is left in the phase if one takes into account, at a fixed energy, data over the whole angular region and uses the unitarity property of the amplitude.[1–6,8]

It is profitable to recall right now in more detail the problem of phase shift analysis at fixed energy, for the case of a reaction between spinless particles. The modulus (squared) of the amplitude $A\,(z = \cos\theta)$ (θ = c.m. scattering angle) is supposed to be known on the physical region $-1 < \cos\theta < 1$, from measurements of the differential cross section

$$d\sigma/d\Omega\,(z) = A\,(z) A^*(z), \quad -1 < z < 1, \tag{1.1}$$

amplitude $A\,(z)$ will vanish at one of these points, but we cannot *a priori* decide at which. There exists thus a twofold ambiguity concerning the location of the zeros of $A\,(z)$, corresponding to each pair (z_i, z_i^*). It is easy to show that if N pairs of zeros are present, we can choose at will any one of the zeros in each pair and construct an amplitude with the correct modulus along (z_-, z_+), analytic in the cut z plane and vanishing precisely at those zeros. If we define

$$\mathscr{M}_1(z) = \frac{\mathscr{M}(z)}{\Pi_{i=1}^{N}(z-z_i)(z-z_i^*)}, \tag{1.3}$$

then a possible $A\,(z)$ is given by

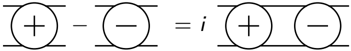
$$A\,(z) = \prod_{j=1}^{N}(z - z_j)\sqrt{\mathscr{M}_1(z)}, \tag{1.4}$$

where the product extends over the given choice of N zeros. There exists thus at least a 2^N ambiguity in the reconstruction of the amplitude, for N distinct pairs of simple zeros. This is the discrete ambiguity "of the zeros."

If the amplitude were a polynomial of degree N, this

Resolving phase-ambiguities: unitarity

$*)$ Assume a scalar reaction in the energy region of <u>elastic unitarity</u>. For the full amplitude $A(s,t)$, unitarity is an integral-constraint:



$$\mathrm{Im}\left[A(s,t)\right] = \frac{\left|\vec{k}\right|}{8\pi\sqrt{s}} \int \frac{d\Omega_k}{4\pi} A(s,\cos\theta_1) A^{*}(s,\cos\theta_2),$$

the integral over $d\Omega_k$ remains from the intermediate phase-space integration.

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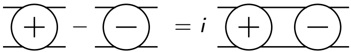

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the integral over $d\Omega_k$ remains from the intermediate phase-space integration.

$*$) As soon as we project to partial waves $A_\ell(s)$, the elastic unitarity-relations become simpler [$\rho(s)$ is a phase-space factor]:

$$\mathrm{Im}\left[A_\ell(s)\right] = \rho(s)\left|A_\ell(s)\right|^2, \text{ or } A_\ell(s) = \tfrac{1}{2i\rho(s)}\left(e^{2i\delta_\ell(s)} - 1\right).$$

People have studied the effect of this p.w.-constraint, on the discrete ambiguities in a TPWA, in the past.

$\rightarrow$ Result: the consensus is that elastic unitarity boils the 2^L discrete ambiguous solutions down to only 2 (!), independently of the order L.

$\hookrightarrow$ 'Crichton-ambiguities' [J. H. Crichton (1966)]
[D. Atkinson, PDF-note, U. Groningen (2002)]

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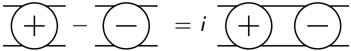

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$\rightarrow$ Result: 2^L ambiguities $\rightarrow$ only 2 (!), independently of L.

$\hookrightarrow$ 'Crichton-ambiguities' [J. H. Crichton (1966)]

$\ast$) Further directions to investigate:

(i) (Re-) derive Crichton's ambiguity starting from the integral-constraint?

(ii) Integral-constraints vs. phase-rotations for multi-channel problems?

Summary

*) Continuum ambiguities ($L \to \infty$) and discrete ambiguities (TPWA at finite L) are in the end manifestations of the same thing: phase-rotations.
$\hookrightarrow$ Although: Structure is richer (i.e. more complicated) for spin-reactions.

*) Spinless case: Only one observable, i.e. σ_0, cannot resolve all discrete ambiguities in a TPWA.

With spin: Polarization observables are capable of resolving discrete ambiguities in a TPWA!
$\to$ complete experiments!

*) It may be worthwhile to do general mathematical studies concerning the restrictions on phase-rotations imposed by:

(i) analyticity: First attempts on analyticity-constraints in one variable were quite pedestrian. Formal mathematical study on application of analyticity in two variables exists: [Stefanescu-paper].

(ii) unitarity: Elastic unitarity in the partial wave basis already studied in quite some detail. Could such results be generalized to more complicated unitarity-relations?

Some references

[A. Gersten (1969)]: A. Gersten, Nucl. Phys. B **12**, p. 537 (1969).

[Bowcock & Burkhardt (1975)]: J. E. Bowcock and H. Burkhardt, Rep. Prog. Phys. **38**, 1099 (1975).

[L. P. Kok (1976)]: L.P. Kok., *Ambiguities in Phase Shift Analysis*, In ∗Delhi 1976, Conference On Few Body Dynamics∗, Amsterdam 1976, 43-46.

[E. Barrelet (1972)]: E. Barrelet, Nuovo Cimento **8A**, 331 (1972).

[Dean & Lee (1972)]: N. W. Dean and P. Lee, Phys. Rev. D **5**, 2741 (1972).

[A. S. Omelaenko (1981)]: A. S. Omelaenko, Sov. J. Nucl. Phys. **34**, 406 (1981).

[I. S. Stefanescu (1984)]: I. S. Stefanescu, J. Math. Phys. **25** (6), 2052 (1984).

Thank You!