The complete experiment problem in a truncated PWA

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Plan for the talk

Consider, for $2 \to 2$-reactions, the two problems of:

(i) Extraction of full reaction-amplitudes,

(ii) Extraction of partial waves up to some cutoff $\ell_{\mathrm{max}} = L$,

using nothing but the measurable observables $\leftrightarrow$ 'complete experiments'.

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Examples with ascending 'spin-complification':

Reactions involving scalar particles (e.g. $\pi\pi$-scattering),

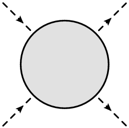
$\Downarrow$

Spin-0 Spin-$\frac{1}{2}$ scattering (e.g. πN-scattering),

$\Downarrow$

Photoproduction of pseudoscalar mesons (e.g. pion-photoproduction).

Extraction of the full amplitude in a reaction of scalars

Consider spin-less $2 \rightarrow 2$-amplitude: $A(W, \theta) =$ 

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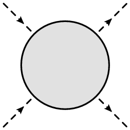
Consider spin-less $2 \to 2$-amplitude: $A(W, \theta) =$ 

$*)$ One single *observable* in scattering experiment: diff. cross section

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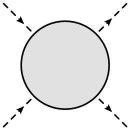
Consider spin-less $2 \to 2$-amplitude: $A(W, \theta) =$ 

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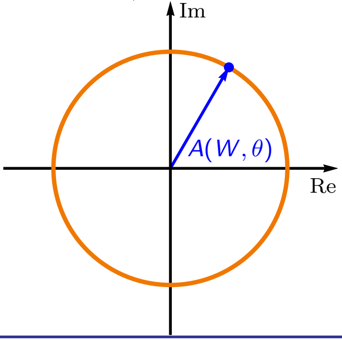

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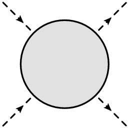
Consider spin-less $2 \to 2$-amplitude: $A(W, \theta) =$ 

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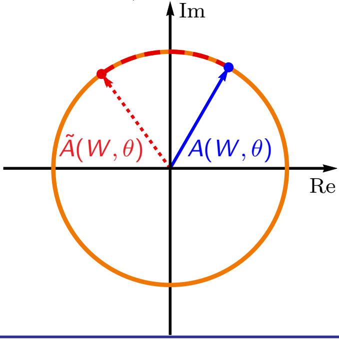
$\hookrightarrow$ Result <u>un</u>changed by multiplication with W- and θ-dependent phase, called <u>'continuum ambiguity'</u>:

$$A(W, \theta) \to \tilde{A}(W, \theta) := e^{i\varphi(W,\theta)} A(W, \theta)$$



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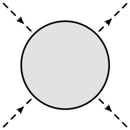
Consider spin-less $2 \rightarrow 2$-amplitude: $A(W, \theta) =$ 

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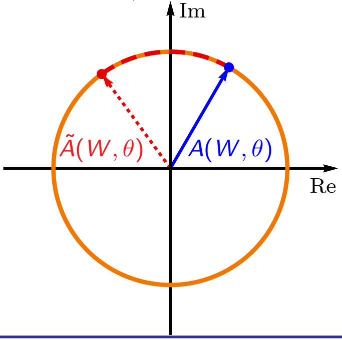

$\Rightarrow$ Implications for partial wave decomposition and in particular for a truncated PWA?

Discrete ambiguities in scalar TPWAs

$*$) A general truncated (i.e. polynomial-) amplitude for arbitrary L, $A = \sum_{\ell=0}^{L} (2\ell + 1) A_{\ell} P_{\ell}(\cos\theta)$, has the linear-factorization:

$$A = \lambda \left(\cos\theta - \alpha_1\right)\left(\cos\theta - \alpha_2\right) \ldots \left(\cos\theta - \alpha_L\right) \text{, with } \lambda \propto A_L .$$

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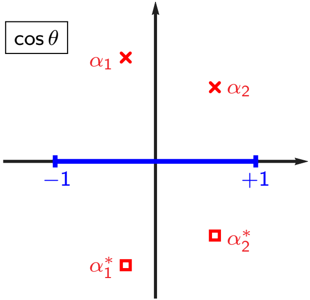
*) roots $(\lambda, \{\alpha_i\}) \leftrightarrow$ partial waves $\{A_{\ell}\}$

*) Define 'mappings' $\boldsymbol{\pi}_n$, which comprise all possibilities to complex conjugate subsets of the roots:

$$\alpha_i \longrightarrow \boldsymbol{\pi}_n \left(\alpha_i\right) = \begin{cases} \alpha_i , \\ \text{or } \alpha_i^{*} , \end{cases} \qquad n = 0, \ldots, 2^L - 1$$

(Example for $L = 2$ on the right $\rightarrow$)

$\hookrightarrow$ There exist in total 2^L possibilities and thus maps $\boldsymbol{\pi}_n$.



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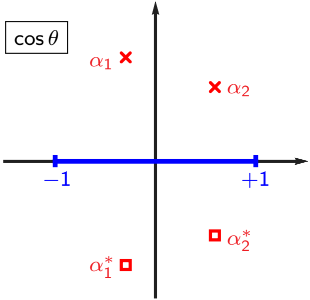

$*$) One can transform to 2^L ambiguous amplitudes:

$$A^{(n)} = \lambda \prod_{i=1}^{L} \left(\cos\theta - \boldsymbol{\pi}_n \left[\alpha_i\right]\right) \equiv \sum_{\ell=0}^{L} (2\ell + 1) A_{\ell}^{(n)}(W) P_{\ell}(\cos\theta),$$

which <u>all</u> have the same c.s. $\sigma_0 = \left|\lambda\right|^2 \prod_{i=1}^{L} \left(\cos\theta - \alpha_i^{*}\right) \left(\cos\theta - \alpha_i\right)$.

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$$e^{i\varphi_n(W,\theta)} = \frac{A^{(n)}(W, \theta)}{A(W, \theta)} = \frac{(\cos\theta - \boldsymbol{\pi}_n[\alpha_1]) \ldots (\cos\theta - \boldsymbol{\pi}_n[\alpha_L])}{(\cos\theta - \alpha_1) \ldots (\cos\theta - \alpha_L)}$$

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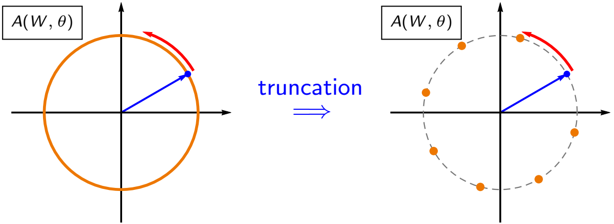
$*$) Illustration: discrete ambiguities are a *remnant* of the continuum ambiguity



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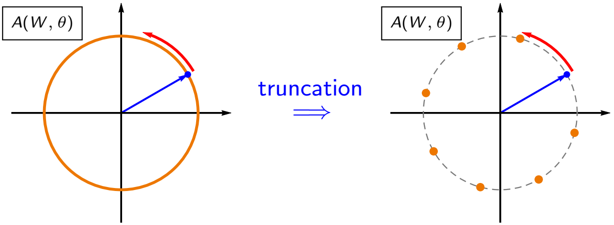

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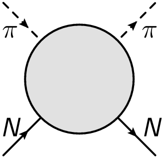
Now: Look at a reaction involving particles with spin!

πN-scattering: amplitudes and observables

Consider the Spin-0 - Spin-$\frac{1}{2}$ scattering amplitude in the CMS:



$$= \mathcal{T}_{fi} = \frac{q}{2\pi}\chi^{\dagger}_{m_{s_f}} \left[G(W,\theta) + i\left(\vec{\sigma}\cdot\hat{\vec{n}}\right) H(W,\theta) \right] \chi_{m_{s_i}}$$

cf. [Hoehler (1983)]

$\rightarrow$ Process fully described by: 'non-spin-flip amplitude' $G\,(W,\theta)$ &
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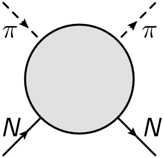

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$*$) A more convenient basis: 'transversity amplitudes'

$$b_1(W,\theta) = \frac{1}{\sqrt{2}}\,(G + i\,H) \text{ \& } b_2(W,\theta) = \frac{1}{\sqrt{2}}\,(G - i\,H).$$

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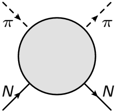

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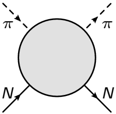
∗) 'Transversity amplitudes': $b_1 = \frac{1}{\sqrt{2}} \left(G + i H \right)$ & $b_2 = \frac{1}{\sqrt{2}} \left(G - i H \right)$.

∗) 4 (polarization) observables can be measured in this reaction:

Observable	Transversity representation	Measurement
σ_0	$\lvert b_1 \rvert^2 + \lvert b_2 \rvert^2$	Unpolarized diff. CS
$\check{P}$	$\lvert b_1 \rvert^2 - \lvert b_2 \rvert^2$	Recoil-polarization
$-\check{R}$	$2 \, \mathrm{Im} \left[b_1^{*} b_2 \right]$	Spin-rotation-
$\check{A}$	$2 \, \mathrm{Re} \left[b_1^{*} b_2 \right]$	parameters

πN-scattering: amplitudes and observables



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Observable	Transversity representation
σ_0	$\left| b_1 \right|^2 + \left| b_2 \right|^2 = \left[\begin{array}{cc} b_1^* & b_2^* \end{array} \right] \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \left[\begin{array}{c} b_1 \\ b_2 \end{array} \right]$
$\check{P}$	$\left| b_1 \right|^2 - \left| b_2 \right|^2 = \left[\begin{array}{cc} b_1^* & b_2^* \end{array} \right] \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right] \left[\begin{array}{c} b_1 \\ b_2 \end{array} \right]$
$-\check{R}$	$2 \,\mathrm{Im} \left[b_1^* b_2 \right] = \left[\begin{array}{cc} b_1^* & b_2^* \end{array} \right] \left[\begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right] \left[\begin{array}{c} b_1 \\ b_2 \end{array} \right]$
$\check{A}$	$2 \,\mathrm{Re} \left[b_1^* b_2 \right] = \left[\begin{array}{cc} b_1^* & b_2^* \end{array} \right] \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \left[\begin{array}{c} b_1 \\ b_2 \end{array} \right]$

Fact: Observables are bilinear forms: $\check{\mathcal{O}}^{\alpha} = \sum_{i,j} b_i^* \sigma_{ij}^{\alpha} b_j$, $\alpha = 0, \ldots, 3$, defined via Pauli-matrices: $\sigma^0 = \mathbb{1}$, $\sigma^1 = \sigma_x$, $\sigma^2 = \sigma_y$, $\sigma^3 = \sigma_z$.

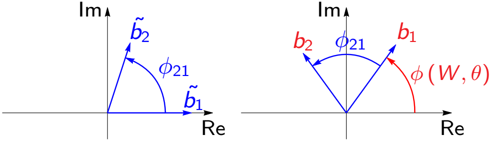
πN-scattering: Complete experiment analysis (CEA)

Goal: Want to extract b_1 & b_2 up to one overall phase, i.e. for

$$b_1 = |b_1| \, e^{i\phi_1} \text{ and } b_2 = |b_2| \, e^{i\phi_2},$$

we want to know the quantities

$$|b_1| \, , \, |b_2| \text{ and } \phi_{21} = \phi_2 - \phi_1 .$$



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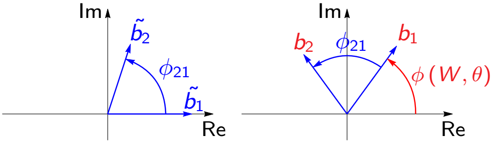

However: 4 observables $\left\{\sigma_0, \check{P}, \check{R}, \check{A}\right\}$ vs. 3 parameters $\left\{|b_1| \, , |b_1| \, , \phi_{21}\right\}$.

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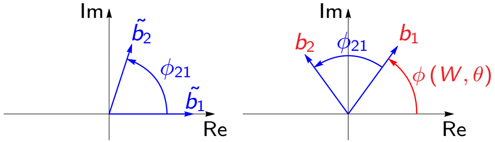

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$\hookrightarrow$ Complete-experiment problem: What are the minimal subsets of the observables $\left\{ \sigma_0, \check{P}, \check{R}, \check{A} \right\}$, which allow for the unique extraction of the amplitudes b_1 & b_2 up to one overall phase?

- $*$) Analysis operates on each bin in (W, θ) individually.
- $*$) Consider idealized (academic) case without measurement uncertainty!

πN-scattering: Complete experiment analysis (CEA) II

$*)$ The 'diagonal' observables: $\sigma_0 = |b_1|^2 + |b_2|^2$, $\check{P} = |b_1|^2 - |b_2|^2$ can be inverted for the two moduli:

$$|b_1| = \sqrt{\frac{1}{2}\left(\sigma_0 + \check{P}\right)},\ |b_2| = \sqrt{\frac{1}{2}\left(\sigma_0 - \check{P}\right)}.$$

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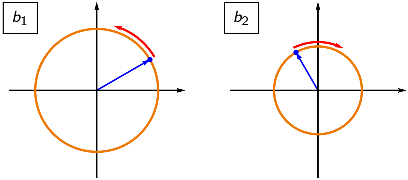
*) σ_0 and $\check{P}$ only constrain the amplitudes to two circles.



$\hookrightarrow$ σ_0 and $\check{P}$ have a '2-fold' continuum-ambiguity:

$b_1(W,\theta) \rightarrow e^{i\varphi_1(W,\theta)} b_1(W,\theta)$ & $b_2(W,\theta) \rightarrow e^{i\varphi_2(W,\theta)} b_2(W,\theta)$.

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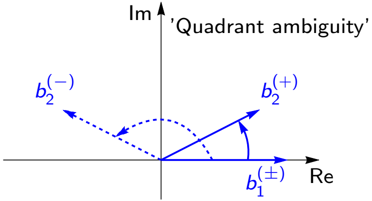
$$-\check{R} = 2\,\mathrm{Im}\left[b_1^* b_2\right] = 2\,|b_1|\,|b_2|\,\mathrm{Im}\left[e^{i\phi_{21}}\right] = 2\,|b_1|\,|b_2|\,\sin\phi_{21},$$

$\Rightarrow \sin\phi_{21} = \frac{-\check{R}}{2\,|b_1||b_2|}$, [Nakayama (2018)]

$$\Rightarrow \phi_{21} \equiv \phi_{21}^{\lambda} = \begin{cases} \alpha_{21} & ,\lambda = +, \\ \pi - \alpha_{21} & ,\lambda = -, \end{cases}$$

α_{21} uniquely defined on $[-\pi/2, \pi/2]$.



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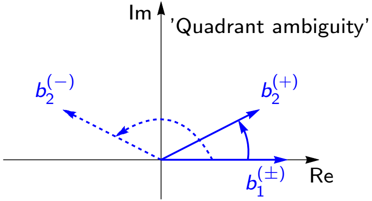


*) $\check{A} = 2\,\mathrm{Re}\left[b_1^* b_2\right] = 2\,|b_1|\,|b_2| \cos\phi_{21}$ can resolve the ambiguity, since:

$$\check{A}^{\pm} = 2\,|b_1|\,|b_2| \cos\phi_{21}^{\pm} = \pm 2\,|b_1|\,|b_2| \cos\alpha_{21} = \pm\check{A}.$$

πN-scattering: Complete experiment analysis (CEA) II

∗) Observables: $\sigma_0 = |b_1|^2 + |b_2|^2$, $\check{P} = |b_1|^2 - |b_2|^2$ yield the two moduli:

$$|b_1| = \sqrt{\frac{1}{2}\left(\sigma_0 + \check{P}\right)},\ |b_2| = \sqrt{\frac{1}{2}\left(\sigma_0 - \check{P}\right)}.$$

∗) σ_0 and $\check{P}$ have a '2-fold' continuum-ambiguity:

$$b_1(W,\theta) \rightarrow e^{i\varphi_1(W,\theta)} b_1(W,\theta)\ \&\ b_2(W,\theta) \rightarrow e^{i\varphi_2(W,\theta)} b_2(W,\theta).$$

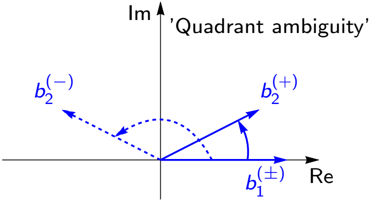
∗) In order to get the relative phase $\phi_{21} = \phi_2 - \phi_1$, take, for instance:

$$-\check{R} = 2\,\mathrm{Im}\left[b_1^* b_2\right] = 2\ |b_1|\,|b_2|\ \mathrm{Im}\left[e^{i\phi_{21}}\right] = 2\ |b_1|\,|b_2|\ \sin\phi_{21},$$

$\Rightarrow \sin\phi_{21} = \frac{-\check{R}}{2\,|b_1||b_2|}$, [Nakayama (2018)]

$$\Rightarrow \phi_{21} \equiv \phi_{21}^{\lambda} = \begin{cases} \alpha_{21} & ,\lambda = +, \\ \pi - \alpha_{21} & ,\lambda = -, \end{cases}$$

α_{21} uniquely defined on $[-\pi/2, \pi/2]$.



↪ 'Complete set of 4 observables':

$$\left\{\sigma_0, \check{P}, \check{R}, \check{A}\right\}.$$

πN-scattering: Truncated partial wave analysis (TPWA)

$*$) Search for 'complete experiments', facilitating the unique extraction of πN-partial waves $T_{\ell\pm}(W)$ ($\leftrightarrow J = |\ell \pm 1/2|$), up to cutoff $\ell_{\mathrm{max}} = L$.

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$$b_1(W,\theta) = \frac{1}{\sqrt{2}}\Big[\sum_{\ell=0}^{L} \left\{ (\ell+1) T_{\ell+}(W) + \ell T_{\ell-}(W) \right\} P_\ell(\cos\theta) + i \sin\theta \sum_{\ell=1}^{L} \left\{ T_{\ell+}(W) - T_{\ell-}(W) \right\} P_\ell^{\prime}(\cos\theta) \Big].$$

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$*$) Linear factorization of the normalized polynomial $\bar{A}_{2L}(t)$:

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$$\text{and } b_2(W,\theta) = b_1(W,-\theta) = \frac{1}{\sqrt{2}} \frac{a_{2L}}{(1+t^2)^L} \prod_{i=1}^{2L} (t+\alpha_i) \, .$$

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πN-scattering: Truncated partial wave analysis (TPWA) II

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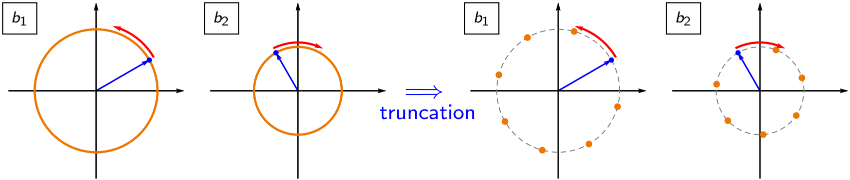
Illustration:



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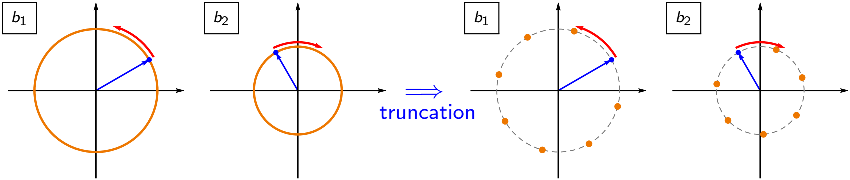

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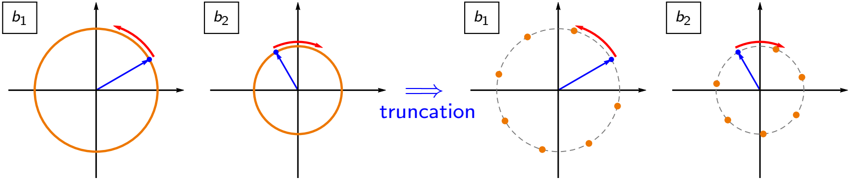

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↪ Observable capable of resolving all the discrete ambiguities:
$\check{R} = -2\,\mathrm{Im}\left[b_1^* b_2\right] = -2\,|b_1|\,|b_2| \sin\phi_{21}$. $\left(\check{A} = 2\,\mathrm{Re}\left[b_1^* b_2\right]\right.$ inv. for $\alpha_i \rightarrow \alpha_i^* \ \forall i.\left.\right)$

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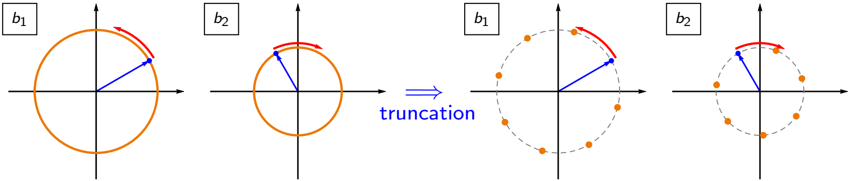

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$\Rightarrow$ Can propose a 'complete set of 3' for the TPWA: $\left\{ \sigma_0, \check{P}, \check{R} \right\}$!

Summary: πN-scattering

CEA

∗) 'Diagonal' observables σ_0 and $\check{P}$ yield the moduli:

$$|b_1| = \sqrt{\frac{1}{2}\left(\sigma_0 + \check{P}\right)}, |b_2| = \sqrt{\frac{1}{2}\left(\sigma_0 - \check{P}\right)}.$$

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TPWA

$\ast$) Truncation at $L \geq 1$ ($t = \tan\theta/2$):

$$b_1 \propto \frac{\prod_i (t-\alpha_i)}{\left(1+t^2\right)^L} \text{ \& } b_2 \propto \frac{\prod_i (t+\alpha_i)}{\left(1+t^2\right)^L}.$$

$\ast$) 2^{2L} discrete ambiguities of σ_0 & $\check{P}$:

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$\hookrightarrow$ $\check{R} = -2\,\mathrm{Im}\left[b_1^* b_2\right] = -2\left|b_1\right|\left|b_2\right|\sin\phi_{21}$ can resolve the discrete ambiguities.

$\hookrightarrow$ 'Complete set of 3': $\left\{\sigma_0, \check{P}, \check{R}\right\}$.

Photoproduction formalism

$*)$ Consider the reaction: $\gamma + N \longrightarrow \pi + N$.

$\hookrightarrow$ Number of spin-amplitudes $N = \underbrace{2}_{\gamma} * \underbrace{2}_{N} * \underbrace{2}_{N} / \underbrace{2}_{\text{Parity}} = 4$.

E.g. transversity amplitudes: $b_1(W, \theta), \ldots, b_4(W, \theta)$.

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$$
\begin{aligned}
b_1 = \frac{1}{2} \Bigg[\frac{1}{\sqrt{2}} \cos\frac{\theta}{2} \Bigg\{ & \sin\theta \sum_{\ell=1}^{L} \left[E_{\ell+} - M_{\ell+} - E_{(\ell+1)-} - M_{(\ell+1)-} \right] \left(P_{\ell}^{''}(x) - P_{\ell+1}^{''}(x) \right) \\
& - i \sum_{\ell=0}^{L} \left[(\ell+2) E_{\ell+} + \ell M_{\ell+} + \ell E_{(\ell+1)-} - (\ell+2) M_{(\ell+1)-} \right] \left(P_{\ell}^{'}(x) - P_{\ell+1}^{'}(x) \right) \Bigg\} \\
& + \frac{1}{\sqrt{2}} \sin\frac{\theta}{2} \Bigg\{ \sum_{\ell=0}^{L} \left[(\ell+2) E_{\ell+} + \ell M_{\ell+} - \ell E_{(\ell+1)-} + (\ell+2) M_{(\ell+1)-} \right] \left(P_{\ell}^{'}(x) + P_{\ell+1}^{'}(x) \right) \\
& + i \sin\theta \sum_{\ell=1}^{L} \left[E_{\ell+} - M_{\ell+} + E_{(\ell+1)-} + M_{(\ell+1)-} \right] \left(P_{\ell}^{''}(x) + P_{\ell+1}^{''}(x) \right) \Bigg\} \Bigg],
\end{aligned}
$$

$b_2 = \ldots$

Photoproduction-observables in the transversity basis

Observable	Transversity representation	Type
σ_0	$\frac{1}{2}\left(\lvert b_1\rvert^2 + \lvert b_2\rvert^2 + \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$	
$\check{\Sigma}$	$\frac{1}{2}\left(-\lvert b_1\rvert^2 - \lvert b_2\rvert^2 + \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$	$\mathcal{S}$
$\check{T}$	$\frac{1}{2}\left(\lvert b_1\rvert^2 - \lvert b_2\rvert^2 - \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$	
$\check{P}$	$\frac{1}{2}\left(-\lvert b_1\rvert^2 + \lvert b_2\rvert^2 - \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$	
$\check{G}$	$\mathrm{Im}\left[-b_1 b_3^* - b_2 b_4^*\right]$	
$\check{H}$	$-\mathrm{Re}\left[b_1 b_3^* - b_2 b_4^*\right]$	$\mathcal{BT}$
$\check{E}$	$-\mathrm{Re}\left[b_1 b_3^* + b_2 b_4^*\right]$	
$\check{F}$	$\mathrm{Im}\left[b_1 b_3^* - b_2 b_4^*\right]$	
$\check{O}_{x'}$	$-\mathrm{Re}\left[-b_1 b_4^* + b_2 b_3^*\right]$	
$\check{O}_{z'}$	$\mathrm{Im}\left[-b_1 b_4^* - b_2 b_3^*\right]$	$\mathcal{BR}$
$\check{C}_{x'}$	$\mathrm{Im}\left[b_1 b_4^* - b_2 b_3^*\right]$	
$\check{C}_{z'}$	$\mathrm{Re}\left[b_1 b_4^* + b_2 b_3^*\right]$	
$\check{T}_{x'}$	$-\mathrm{Re}\left[-b_1 b_2^* + b_3 b_4^*\right]$	
$\check{T}_{z'}$	$-\mathrm{Im}\left[b_1 b_2^* - b_3 b_4^*\right]$	$\mathcal{TR}$
$\check{L}_{x'}$	$-\mathrm{Im}\left[-b_1 b_2^* - b_3 b_4^*\right]$	
$\check{L}_{z'}$	$\mathrm{Re}\left[-b_1 b_2^* - b_3 b_4^*\right]$	

∗) Bilinear forms:

$$\check{\mathcal{O}}^{\alpha} = \frac{1}{2} \sum_{i,j} b_i^* \tilde{\Gamma}^{\alpha}_{ij} b_j;$$

4×4 Dirac-matrices $\tilde{\Gamma}^{\alpha}$.

∗) The 'diagonal' observables are

$$\left\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\right\}.$$

∗) $\mathcal{BT}$, $\mathcal{BR}$ and $\mathcal{TR}$ are of 'non-diagonal' type.

Summary of results: Photoproduction

CEA

TPWA

∗) 'Diagonal' observables $\left\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\right\}$ yield the moduli:

$$|b_1|, \; |b_2|, \; |b_3| \; \& \; |b_4|.$$

∗) $\left\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\right\}$ invariant under '4-fold' continuum ambiguity:

$$b_j(W, \theta) \rightarrow e^{i \varphi_j(W,\theta)} b_j(W, \theta), \; j = 1, \ldots, 4.$$

∗) 4 additional observables can uniquely fix the relative-phases,

e.g.: ϕ_{21}, ϕ_{32} and ϕ_{43}.

cf. [Chiang/Tabakin (1997)] & [Nakayama (2018)]

↪ Complete sets of (at least) 8 observables, e.g.:

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Summary of results: Photoproduction

CEA

$*)$ 'Diagonal' observables $\left\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\right\}$ yield the moduli:

$$|b_1|, \ |b_2|, \ |b_3| \ \& \ |b_4|.$$

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TPWA

$*)$ Truncation at $L \geq 1$ ($t = \tan\theta/2$):

$$b_1 \propto \frac{\prod_i (t + \beta_i)}{(1+t^2)^L} \ \& \ b_2 \propto \frac{\prod_i (t - \beta_i)}{(1+t^2)^L},$$

$$b_3 \propto \frac{\prod_i (t + \alpha_i)}{(1+t^2)^L} \ \& \ b_4 \propto \frac{\prod_i (t - \alpha_i)}{(1+t^2)^L}.$$

[A.S. Omelaenko (1981)]

$*)$ 2^{4L} discr. amb.'s of $\left\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\right\}$:

$$\alpha_i \rightarrow \boldsymbol{\pi}_n(\alpha_i),$$

$$\beta_i \rightarrow \boldsymbol{\pi}_n(\beta_i), \ n = 0, \ldots, 2^{4L} - 1.$$

$*)$ Discr. ambiguities: 4-fold rotations,

$$b_j(W,\theta) \rightarrow e^{i\varphi_j^n(W,\theta)} b_j(W,\theta), \ n = 0, .., 2^{4L} - 1.$$

$\hookrightarrow$ Observables capable of resolving all discr. ambiguities (only 1 needed):

$\check{F}$, $\check{G}$, any $\mathcal{BR}$, any $\mathcal{TR}$.

$\hookrightarrow$ 'Complete sets of 5', e.g.:

$$\left\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}, \check{F}\right\}.$$

Conclusion and generalization

For a $2 \to 2$ reaction involving particles with spin: N (transversity) amplitudes b_i vs. N^2 polarization observables $\check{\mathcal{O}}^{\alpha} = \langle b | \Gamma^{\alpha} | b \rangle$.

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*) TPWA: truncate the partial-wave expansion at a certain $L \geq 1$

$\hookrightarrow$ $n_{\text{p.w.}} * L$ partial waves, or $N * L$ roots, to be determined.

The N 'diagonal observables'

$$\check{\mathcal{O}}^{\alpha}_{\boldsymbol{d}}(W,\theta) = \pm \left| b_1(W,\theta) \right|^2 \pm \ldots \pm \left| b_N(W,\theta) \right|^2,$$

have 2^{N*L} discrete ambiguities, generated by N-fold rotations:

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However: the $N(N-1)$ 'non-diagonal' observables

$$\check{\mathcal{O}}^{\alpha}_{\boldsymbol{nd}}(W, \theta) = \sum_{i,j=1}^{N} \Gamma^{\alpha}_{ij} \, b_i^{*}(W, \theta) \, b_j(W, \theta),$$

can (in principle) resolve such ambiguities.

$\to$ Can motivate 'complete sets of $N + 1$ observables' algebraically.

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Thank You!