Determining dominant partial waves in photoproduction via moment analysis

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Introduction

Methods used to extract resonance- (/partial wave-) content from data:

$*$) Energy-dependent (ED) PWA:

$$K_{ab} = \sum_j \frac{g_j^a g_j^b}{s - m_j^2} + f_{ab}(s).$$

$\rightarrow$ Currently accepted method to extract pole-parameters

$\rightarrow$ Very complicated models/codes; years of work to build a model; dedicated groups of experts

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→ Very complicated models/codes; years of work to build a model; dedicated groups of experts

*) Energy-independent/single-energy (SE) PWA:

$$\sigma_0(W, \theta) = \frac{q}{k} \sum_{n=0}^{2\ell_{\mathrm{max}}} a_n^{\sigma_0}(W) P_n(\cos\theta), a_n^{\sigma_0}(W) = \sum_{\ell, k=0}^{\ell_{\mathrm{max}}} A_\ell^*(W) \mathcal{C}_{\ell k}^n A_k(W).$$

→ Simpler procedure; Can yield sensible results for partial waves, but does not have to, because of:

→ Ambiguities!!; Furthermore, for reactions with spin, this still takes some coding...

Introduction

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$$\boxed{\sigma_0(W,\theta) = \frac{q}{k} \sum_{n=0}^{2\ell_{\mathrm{max}}} a_n^{\sigma_0}(W) P_n(\cos\theta),} \; a_n^{\sigma_0}(W) = \sum_{\ell,k=0}^{\ell_{\mathrm{max}}} A_\ell^*(W) \mathcal{C}_{\ell k}^n A_k(W)$$

$*)$ Is there an even simpler method to learn first lessons about newly measured data?

$\longrightarrow$ Yes: moment-analysis (we call this "ℓ_{max}-fit", in our group)

$\hookrightarrow$ Illustrate moment-analysis on the example of photoproduction of pions $\gamma p \longrightarrow \pi^0 p$ and eta mesons $\gamma p \longrightarrow \eta p$ in the following ...

From the multipole expansion to polynomial observables

$*$) Photoproduction is described by 4 CGLN-amplitudes F_i, with expansions into electric ($E_{\ell\pm}$) and magnetic ($M_{\ell\pm}$) multipoles:

$$F_1(W,\theta) = \sum_{\ell=0}^{\infty} \Big\{ \left[\ell M_{\ell+}(W) + E_{\ell+}(W)\right] P^{'}_{\ell+1}(x) + \left[(\ell+1) M_{\ell-}(W) + E_{\ell-}(W)\right] P^{'}_{\ell-1}(x) \Big\},$$

$$F_2(W,\theta) = \sum_{\ell=1}^{\infty} \left[(\ell+1) M_{\ell+}(W) + \ell M_{\ell-}(W)\right] P^{'}_{\ell}(x),$$

$$F_3(W,\theta) = \sum_{\ell=1}^{\infty} \Big\{ \left[E_{\ell+}(W) - M_{\ell+}(W)\right] P^{''}_{\ell+1}(x) + \left[E_{\ell-}(W) + M_{\ell-}(W)\right] P^{''}_{\ell-1}(x) \Big\},$$

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$$F_4(W,\theta) = \sum_{\ell=2}^{\not\infty\,\ell_{\mathrm{max}}} \left[M_{\ell+}(W) - E_{\ell+}(W) - M_{\ell-}(W) - E_{\ell-}(W)\right] P_{\ell}^{''}(x).$$

$\rightarrow$ For a <u>truncation</u> at ℓ_{max}, the polynomial-orders in $\cos\theta$ are:

$F_1 \sim (\cos\theta)^{\ell_{\mathrm{max}}}$, $F_2 \sim (\cos\theta)^{\ell_{\mathrm{max}}-1}$, $F_3 \sim (\cos\theta)^{\ell_{\mathrm{max}}-1}$, $F_4 \sim (\cos\theta)^{\ell_{\mathrm{max}}-2}$.

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$\rightarrow$ Example: helicity asymmetry $\check{E} = \sigma_0 E = \sigma^{(1/2)} - \sigma^{(3/2)}$

$$\check{E} = \frac{q}{k} \mathrm{Re}\Big[\left|F_1\right|^2 + \left|F_2\right|^2 - 2\cos\theta\, F_1^* F_2 + \sin^2\theta \left\{ F_4^* F_1 + F_3^* F_2 \right\} \Big].$$

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$$\check{E} = \frac{q}{k} \mathrm{Re}\Big[\underbrace{|F_1|^2}_{\sim\cos\theta^{2\ell_{\max}}} + \underbrace{|F_2|^2}_{\sim\cos\theta^{2\ell_{\max}-2}} - 2\cos\theta \underbrace{F_1^* F_2}_{\sim\cos\theta^{2\ell_{\max}-1}} + \underbrace{\sin^2\theta}_{\sim\cos^2\theta} \Big\{ \underbrace{F_4^* F_1}_{\sim\cos\theta^{2\ell_{\max}-2}} + \underbrace{F_3^* F_2}_{\sim\cos\theta^{2\ell_{\max}-2}} \Big\} \Big].$$

Therefore: $\check{E} \sim (\cos\theta)^{2\ell_{\max}}$

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Therefore: $\check{E} \sim (\cos\theta)^{2\ell_{\mathrm{max}}}$

$\ast$) Multiple possible choices of bases for polynomial-expansion, for instance:

(i) $\cos\theta$-monomials: $\check{E}(W,\theta) = \frac{q}{k}\sum_{n=0}^{2\ell_{\mathrm{max}}} \left(a_{\ell_{\mathrm{max}}}\right)_n^{\check{E}}(W)\ \cos\theta^n$,

(ii) (Assoc.) Legendre-poly.'s: $\check{E}(W,\theta) = \frac{q}{k}\sum_{n=0}^{2\ell_{\mathrm{max}}} \left(a_{\ell_{\mathrm{max}}}\right)_n^{\check{E}}(W)\, P_n(\cos\theta)$.

Parametrizations for all polarization observables

$*)$ 16 polarization-observables measurable in photoproduction:

Beam		Target			Recoil			Target + Recoil			
	-	-	-	-	x'	y'	z'	x'	x'	z'	z'
	-	x	y	z	-	-	-	x	z	x	z
unpolarized	$\left(\frac{d\sigma}{d\Omega}\right)_0 = \sigma_0$		$\check{T}$			$\check{P}$		$\check{T}_{x'}$	$\check{L}_{x'}$	$\check{T}_{z'}$	$\check{L}_{z'}$
linear	$\check{\Sigma}$	$\check{H}$	$\check{P}$	$\check{G}$	$\check{O}_{x'}$	$\check{T}$	$\check{O}_{z'}$				
circular		$\check{F}$		$\check{E}$	$\check{C}_{x'}$		$\check{C}_{z'}$				

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linear	$\check{\Sigma}$	$\check{H}$	$\check{P}$	$\check{G}$	$\check{O}_{x'}$	$\check{T}$	$\check{O}_{z'}$				
circular		$\check{F}$		$\check{E}$	$\check{C}_{x'}$		$\check{C}_{z'}$				

$\ast$) Helicity-asymmetry $\check{E}$ from previous example is bilinear in the F_i:

$$\check{E} = \frac{q}{k} \mathrm{Re}\left[\left|F_1\right|^2 + \left|F_2\right|^2 - 2\cos\theta\, F_1^\ast F_2 + \sin^2\theta \left\{ F_4^\ast F_1 + F_3^\ast F_2 \right\} \right]$$

$$= \frac{1}{2}\frac{q}{k} \left[\begin{array}{cccc} F_1^\ast & F_2^\ast & F_3^\ast & F_4^\ast \end{array} \right] \left[\begin{array}{cccc} 2 & -2\cos(\theta) & 0 & \sin(\theta)^2 \\ -2\cos(\theta) & 2 & \sin(\theta)^2 & 0 \\ 0 & \sin(\theta)^2 & 0 & 0 \\ \sin(\theta)^2 & 0 & 0 & 0 \end{array} \right] \left[\begin{array}{c} F_1 \\ F_2 \\ F_3 \\ F_4 \end{array} \right] .$$

Parametrizations for all polarization observables

$\ast$) <u>16 polarization-observables</u> measurable in photoproduction:

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	-	x	y	z	-	-	-	x	z	x	z
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circular		$\check{F}$		$\check{E}$	$\check{C}_{x'}$		$\check{C}_{z'}$				

$\ast$) The same is true for <u>any</u> other observable $\check{\Omega}^{\alpha}(W, \theta)$:

$$\check{\Omega}^{\alpha} = \frac{1}{2}\frac{q}{k}\sin^{\beta_{\alpha}}\theta \begin{bmatrix} F_1^{*} & \dots & F_4^{*} \end{bmatrix} \begin{bmatrix} & & \\ & \hat{A}^{\alpha}(\cos\theta) & \\ & & \end{bmatrix} \begin{bmatrix} F_1 \\ \vdots \\ F_4 \end{bmatrix}$$

$$= \frac{1}{2}\frac{q}{k}\sin^{\beta_{\alpha}}\theta \left\langle F \right| \hat{A}^{\alpha}(\cos\theta) \left| F \right\rangle , \ \alpha = 1, \dots, 16,$$

with $\hat{A}^{\alpha}$ some hermitean 4×4-matrix.

Parametrizations for all polarization observables

*) Observables are bilinear: $\check{\Omega}^{\alpha} = \frac{1}{2} \frac{q}{k} \sin^{\beta_{\alpha}} \theta \left< F \right| \hat{A}^{\alpha}(\cos\theta) \left| F \right>$.

*) Similar counting as in the $\check{\Sigma}$-example leads to Legendre-expansions for all 16 observables, with orders varying as $2\ell_{\mathrm{max}}$:

$$\check{\Omega}^{\alpha}(W,\theta) = \frac{q}{k} \sum_{n=\beta_{\alpha}}^{2\ell_{\mathrm{max}}+\beta_{\alpha}+\gamma_{\alpha}} \left(a_{\ell_{\mathrm{max}}}\right)_{n}^{\check{\Omega}^{\alpha}} (W)\, P_{n}^{\beta_{\alpha}}(\cos\theta).$$

Type	$\check{\Omega}^{\alpha}$	β_{α}	γ_{α}	Type	$\check{\Omega}^{\alpha}$	β_{α}	γ_{α}
	$\sigma_0(\theta)$	0	0		$\check{O}_{x'}$	1	0
$\mathcal{S}$	$\check{\Sigma}$	2	-2	$\mathcal{BR}$	$\check{O}_{z'}$	2	-1
	$\check{T}$	1	-1		$\check{C}_{x'}$	1	0
	$\check{P}$	1	-1		$\check{C}_{z'}$	0	$+1$
	$\check{E}$	0	0		$\check{T}_{x'}$	2	-1
$\mathcal{BT}$	$\check{G}$	2	-2	$\mathcal{TR}$	$\check{T}_{z'}$	1	0
	$\check{H}$	1	-1		$\check{L}_{x'}$	1	0
	$\check{F}$	1	-1		$\check{L}_{z'}$	0	$+1$

Legendre coefficients in terms of multipoles

Example: $$\check{E} \propto (a_2)_0^{\check{E}} \, P_0(\cos\theta) + (a_2)_1^{\check{E}} \, P_1(\cos\theta) + (a_2)_2^{\check{E}} \, P_2(\cos\theta) + (a_2)_3^{\check{E}} \, P_3(\cos\theta) + (a_2)_4^{\check{E}} \, P_4(\cos\theta) \text{ , i.e. } \ell_{\mathrm{max}} = 2;$$

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$$(a_2)_0^{\check{E}} = |E_{0+}|^2 + |M_{1-}|^2 - E_{2-}^* \left(E_{2-} + 3M_{2-}\right) + 3M_{2-}^* \left(M_{2-} - E_{2-}\right) + 3E_{1+}^* \left(E_{1+} + M_{1+}\right) + 3M_{1+}^* E_{1+} + 6E_{2+}^*(E_{2+} + 2M_{2+}) + 12M_{2+}^* E_{2+} - |M_{1+}|^2 - 3\,|M_{2+}|^2$$

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$$
\begin{aligned}
(a_2)_0^{\check{E}} &= |E_{0+}|^2 + |M_{1-}|^2 - E_{2-}^* \left(E_{2-} + 3M_{2-}\right) + 3M_{2-}^* \left(M_{2-} - E_{2-}\right) + 3E_{1+}^* \left(E_{1+} + M_{1+}\right) \\
&\quad + 3M_{1+}^* E_{1+} + 6E_{2+}^* (E_{2+} + 2M_{2+}) + 12M_{2+}^* E_{2+} - |M_{1+}|^2 - 3\,|M_{2+}|^2 \\
&= \begin{bmatrix} E_{0+}^* & E_{1+}^* & \dots & M_{2-}^* \end{bmatrix}
\left[\begin{array}{c|ccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 6 & 0 & 12 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -3 \\
0 & 0 & 0 & 0 & 12 & 0 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & -3 & 0 & 3
\end{array}\right]
\begin{bmatrix} E_{0+} \\ E_{1+} \\ M_{1+} \\ M_{1-} \\ E_{2+} \\ E_{2-} \\ M_{2+} \\ M_{2-} \end{bmatrix}
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Legendre coefficients in terms of multipoles

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$$(a_2)_0^{\check{E}} = |E_{0+}|^2 + |M_{1-}|^2 - E_{2-}^* (E_{2-} + 3M_{2-}) + 3M_{2-}^* (M_{2-} - E_{2-}) + 3E_{1+}^* (E_{1+} + M_{1+}) + 3M_{1+}^* E_{1+} + 6E_{2+}^*(E_{2+} + 2M_{2+}) + 12M_{2+}^* E_{2+} - |M_{1+}|^2 - 3\,|M_{2+}|^2$$

$$= \left[\begin{array}{cccc} E_{0+}^* & E_{1+}^* & \ldots & M_{2-}^* \end{array}\right] \left[\begin{array}{c|ccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 6 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 12 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & 3 \end{array}\right] \left[\begin{array}{c} E_{0+} \\ E_{1+} \\ M_{1+} \\ M_{1-} \\ E_{2+} \\ E_{2-} \\ M_{2+} \\ M_{2-} \end{array}\right]$$

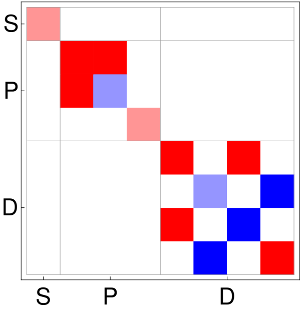
$$= \langle \mathcal{M}_\ell | \, \mathcal{C}_0^{\check{E}} \, | \mathcal{M}_\ell \rangle \equiv \langle S, S \rangle + \langle P, P \rangle + \langle D, D \rangle$$

Generally: $(a_{\ell_{\max}})_k^{\check{\Omega}^\alpha}$ defined by matrices with $\langle \ell_1, \ell_2 \rangle$-interference blocks

Colored ("chessboard"-) plots for interference-blocks

Using the previous example of $\underline{(a_2)_0^{\check{E}}} = \langle \mathcal{M}_\ell | \, \mathcal{C}_0^{\check{E}} \, | \mathcal{M}_\ell \rangle$ for $\ell_{\max} = 2$:

$$\mathcal{C}_0^{\check{E}} = \left[\begin{array}{c|ccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 6 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 12 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & 3 \end{array} \right] \equiv$$



<u>For every matrix-entry:</u>

red = positive number vs. blue = negative number,

& strength of shading $\propto$ magnitude of number.

$\hookrightarrow$ Use color-plots in order to <u>depict block-structure</u> of Legendre-moments.

Two aspects of moment-analysis

I.) ℓ_{max}-analysis

$\ast$) Fit angular distribution using

$$\check{\Omega}^{\alpha} = \frac{q}{k} \sum_{n=\beta_{\alpha}}^{2\ell_{\mathrm{max}}+\beta_{\alpha}+\gamma_{\alpha}} \left(a_{\ell_{\mathrm{max}}}\right)_{n}^{\check{\Omega}^{\alpha}} P_{n}^{\beta_{\alpha}}(\cos\theta),$$

for different ℓ_{max}.

$\ast$) Compare χ^2/ndf for different fits; if unsatisfactory: increase ℓ_{max}

$\hookrightarrow$ Good fit $\rightarrow$ ℓ_{max}-estimate

$\hookrightarrow$ Plot χ^2/ndf vs. energy for all fits $\rightarrow$ "bumps"

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II.) Interpretation

*) Compare fitted $\left(a_{\ell_{\mathrm{max}}}\right)_{n}^{\check{\Omega}^{\alpha}}$ to

$$\left(a_{\ell_{\mathrm{max}}}\right)_{n}^{\check{\Omega}^{\alpha}} = \left\langle \mathcal{M}_{\ell} \right| \mathcal{C}_{n}^{\check{\Omega}^{\alpha}} \left| \mathcal{M}_{\ell} \right\rangle,$$

with model-multipoles $\mathcal{M}_{\ell}$.

*) Calculate $\left(a_{\ell_{\mathrm{max}}}\right)_{n}^{\check{\Omega}^{\alpha}}$ "switching on/off" certain partial waves

↪ Get information on which interferences are important

*) In particular: $\left(a_{\ell_{\mathrm{max}}}\right)_{n_{\mathrm{max}}} = \left\langle \ell_{\mathrm{max}}, \ell_{\mathrm{max}} \right\rangle$.

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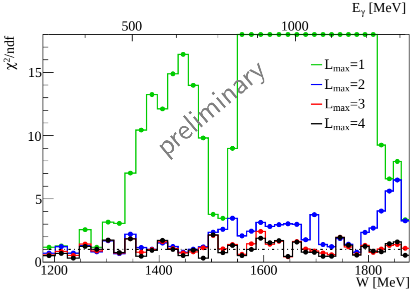
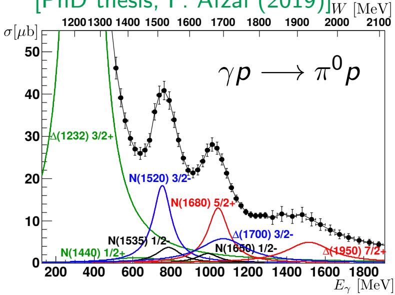
Now: Consider examples of moment analyses for $\gamma p \rightarrow \pi^0 p$ and $\gamma p \rightarrow \eta p$.

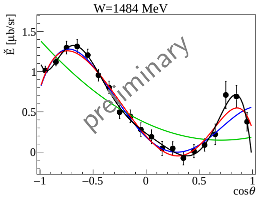
$\check{E}_{\mathrm{A2}}$-data in $\gamma p \longrightarrow \pi^0 p$: ℓ_{max}-analysis

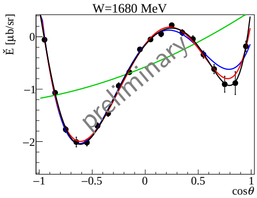
$$\check{E}(W,\theta) = \sigma^{(1/2)} - \sigma^{(3/2)} = \frac{q}{k} \sum_{n=0}^{2\ell_{\mathrm{max}}} \left(a_{\ell_{\mathrm{max}}}\right)_n^{\check{E}} (W)\, P_n(\cos\theta)$$

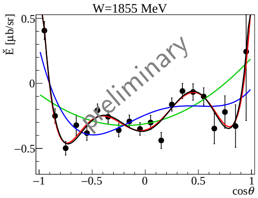
[PhD thesis, F. Afzal (2019)]







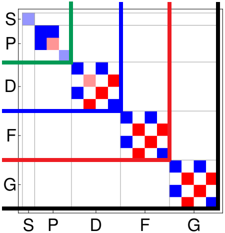


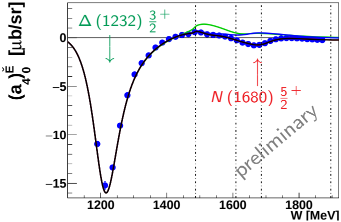


$\check{E}_{\mathrm{A2}}$-data in $\gamma p \longrightarrow \pi^0 p$: moments

$$\check{E}(W,\theta) = \frac{q}{k} \sum_{n=0}^{2\ell_{\mathrm{max}}} \left(a_{\ell_{\mathrm{max}}}\right)_n^{\check{E}}(W)\, P_n(\cos\theta),$$ data: [PhD, F. Afzal (2019)]

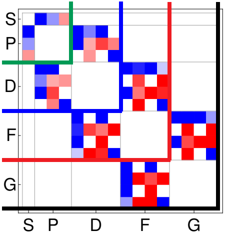
$\mathcal{C}_0^{\check{E}} \equiv$

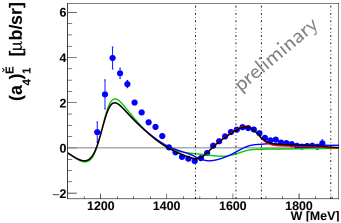




$$(a_4)_0^{\check{E}} = \langle S,S\rangle + \langle P,P\rangle + \langle D,D\rangle + \langle F,F\rangle + \langle G,G\rangle$$

$\mathcal{C}_1^{\check{E}} \equiv$





$$(a_4)_1^{\check{E}} = \langle S,P\rangle + \langle P,D\rangle + \langle D,F\rangle + \langle F,G\rangle$$

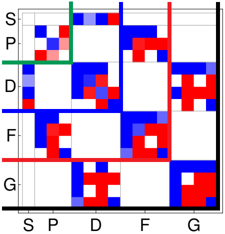
$*$) BnGa 2017_02: green $= S + P$ waves, blue $= S + P + D$ waves,

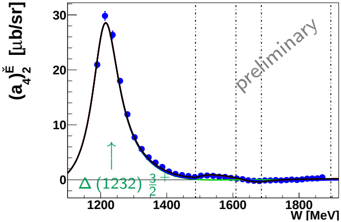
red $= S + P + D + F$ waves, black $= S + P + D + F + G$ waves.

$\check{E}_{\mathrm{A2}}$-data in $\gamma p \longrightarrow \pi^0 p$: moments

$\check{E}(W,\theta) = \frac{q}{k} \sum_{n=0}^{2\ell_{\max}} \left(a_{\ell_{\max}}\right)_n^{\check{E}} (W)\, P_n(\cos\theta)$, data: [PhD, F. Afzal (2019)]

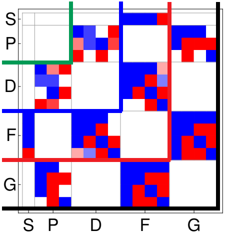
$\mathcal{C}_2^{\check{E}} \equiv$

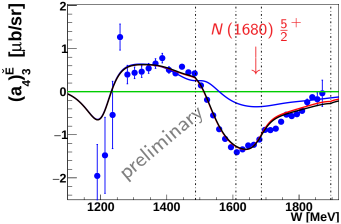




$(a_4)_2^{\check{E}} = \langle P, P\rangle + \langle S, D\rangle + \langle D, D\rangle + \langle P, F\rangle + \langle F, F\rangle + \langle D, G\rangle + \langle G, G\rangle$

$\mathcal{C}_3^{\check{E}} \equiv$





$(a_4)_3^{\check{E}} = \langle P, D\rangle + \langle S, F\rangle + \langle D, F\rangle + \langle P, G\rangle + \langle F, G\rangle$

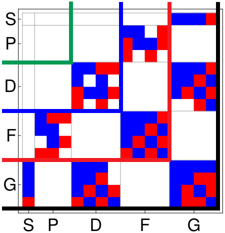
$*$) BnGa 2017_02: green $= S + P$ waves, blue $= S + P + D$ waves,

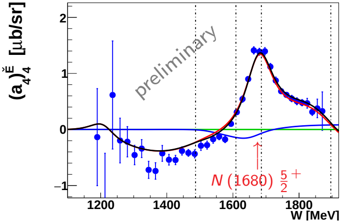
red $= S + P + D + F$ waves, black $= S + P + D + F + G$ waves.

$\check{E}_{\mathrm{A2}}$-data in $\gamma p \longrightarrow \pi^0 p$: moments

$\check{E}(W,\theta) = \frac{q}{k} \sum_{n=0}^{2\ell_{\max}} \left(a_{\ell_{\max}}\right)_n^{\check{E}}(W)\, P_n(\cos\theta)$, data: [PhD, F. Afzal (2019)]

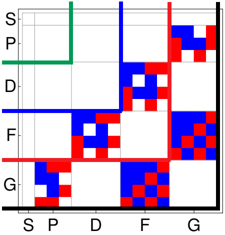
$\mathcal{C}_4^{\check{E}} \equiv$

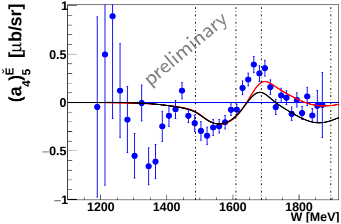




$(a_4)_4^{\check{E}} = \langle D, D\rangle + \langle P, F\rangle + \langle F, F\rangle + \langle S, G\rangle + \langle D, G\rangle + \langle G, G\rangle$

$\mathcal{C}_5^{\check{E}} \equiv$





$(a_4)_5^{\check{E}} = \langle D, F\rangle + \langle P, G\rangle + \langle F, G\rangle$

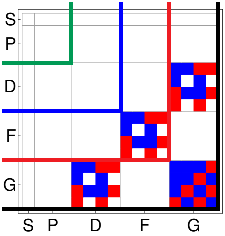
$\ast$) BnGa 2017_02: green = $S + P$ waves, blue = $S + P + D$ waves,

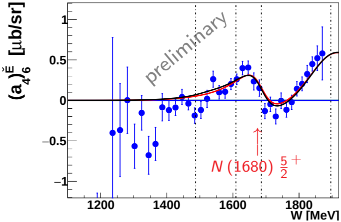
red = $S + P + D + F$ waves, black = $S + P + D + F + G$ waves.

$\check{E}_{\mathrm{A2}}$-data in $\gamma p \longrightarrow \pi^0 p$: moments

$\check{E}(W,\theta) = \frac{q}{k} \sum_{n=0}^{2\ell_{\max}} \left(a_{\ell_{\max}}\right)_n^{\check{E}} (W)\, P_n(\cos\theta)$, data: [PhD, F. Afzal (2019)]

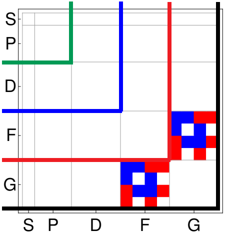
$\mathcal{C}_6^{\check{E}} \equiv$

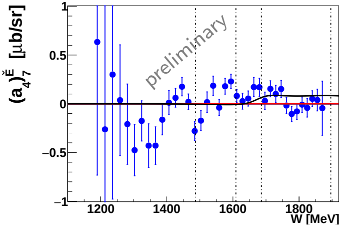




$(a_4)_6^{\check{E}} = \langle F, F \rangle + \langle D, G \rangle + \langle G, G \rangle$

$\mathcal{C}_7^{\check{E}} \equiv$





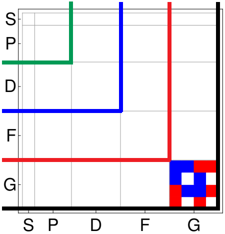
$(a_4)_7^{\check{E}} = \langle F, G \rangle$

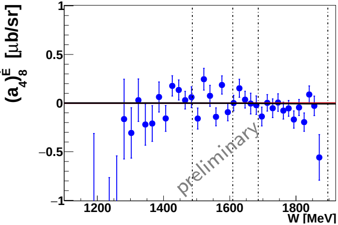
$*$) BnGa 2017_02: green $= S + P$ waves, blue $= S + P + D$ waves,

red $= S + P + D + F$ waves, black $= S + P + D + F + G$ waves.

$\check{E}_{\mathrm{A2}}$-data in $\gamma p \longrightarrow \pi^0 p$: moments

$\check{E}(W,\theta) = \frac{q}{k} \sum_{n=0}^{2\ell_{\max}} \left(a_{\ell_{\max}}\right)_n^{\check{E}} (W)\, P_n(\cos\theta)$,   data: [PhD, F. Afzal (2019)]

$\mathcal{C}_8^{\check{E}} \equiv$ 



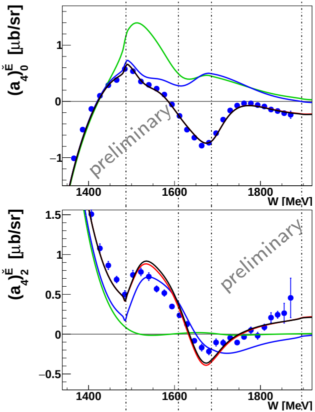
$(a_4)_8^{\check{E}} = \langle G, G \rangle$

This was an example of a dataset which is quite well in agreement with well-established parts of the N^*-spectrum.

$\hookrightarrow$ Can we see more interesting things apart from 'bumps' in the Legendre-moments?

$*$) BnGa 2017_02: green $= S + P$ waves, blue $= S + P + D$ waves,
red $= S + P + D + F$ waves, black $= S + P + D + F + G$ waves.

$\check{E}_{\mathrm{A2}}$-data: 'kinks' in selected Legendre moments



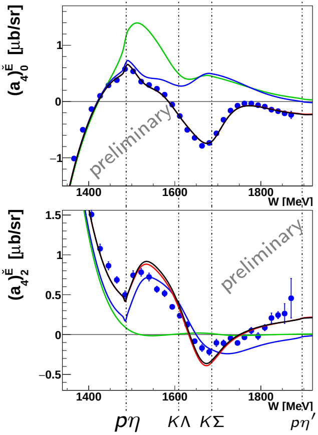
$$(a_4)_0^{\check{E}} = \langle S, S\rangle + \langle P, P\rangle + \langle \boldsymbol{D}, \boldsymbol{D}\rangle + \langle F, F\rangle + \langle G, G\rangle$$

$$(a_4)_2^{\check{E}} = \langle P, P\rangle + \langle \boldsymbol{S}, \boldsymbol{D}\rangle + \langle \boldsymbol{D}, \boldsymbol{D}\rangle + \langle P, F\rangle + \langle F, F\rangle + \langle D, G\rangle + \langle G, G\rangle$$

multipoles: BnGa 2017_02

$*$) Consider sudden changes in direction in fit results for $(a_4)_0^{\check{E}}$ and $(a_4)_2^{\check{E}}$

$\check{E}_{\mathrm{A2}}$-data: 'kinks' in selected Legendre moments



$$(a_4)_0^{\check{E}} = \langle S, S \rangle + \langle P, P \rangle + \langle \boldsymbol{D}, \boldsymbol{D} \rangle + \langle F, F \rangle + \langle G, G \rangle$$

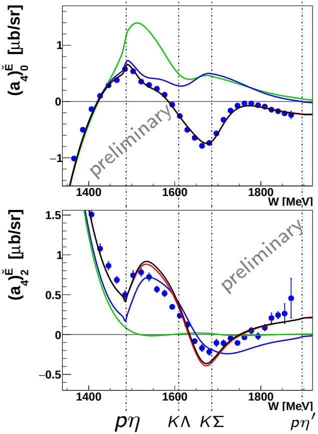
$$(a_4)_2^{\check{E}} = \langle P, P \rangle + \langle \boldsymbol{S}, \boldsymbol{D} \rangle + \langle \boldsymbol{D}, \boldsymbol{D} \rangle + \langle P, F \rangle + \langle F, F \rangle + \langle D, G \rangle + \langle G, G \rangle$$

multipoles: BnGa 2017_02

$*$) Effect occurs at energy of the $p\eta$-threshold

$\hookrightarrow$ 'Cusp'-effect known in ED models / S-Matrix Theory, here visible in a Legendre-moment

$\check{E}_{\mathrm{A2}}$-data: 'kinks' in selected Legendre moments



$$(a_4)_0^{\check{E}} = \langle S, S \rangle + \langle P, P \rangle + \langle \boldsymbol{D}, \boldsymbol{D} \rangle + \langle F, F \rangle + \langle G, G \rangle$$

$$(a_4)_2^{\check{E}} = \langle P, P \rangle + \langle \boldsymbol{S}, \boldsymbol{D} \rangle + \langle \boldsymbol{D}, \boldsymbol{D} \rangle + \langle P, F \rangle + \langle F, F \rangle + \langle D, G \rangle + \langle G, G \rangle$$

multipoles: BnGa 2017_02

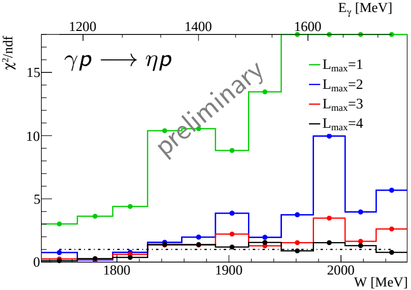
$*$) Parametrization of energy-dependence due to cusps important for the correct determination of resonance parameters

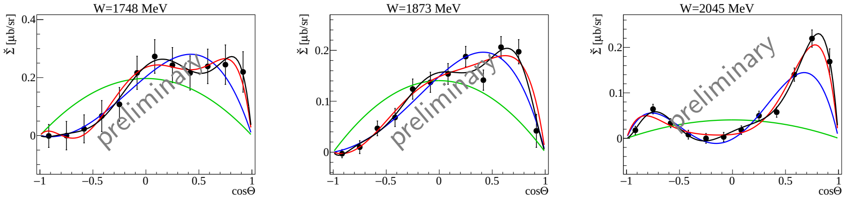
$\rightarrow$ Legendre-moments are quite sensitive to such effects, ideal for comparisons

$\check{\Sigma}_{\mathrm{CBELSA}}$-data in $\gamma p \longrightarrow \eta p$: ℓ_{max}-analysis

$$\check{\Sigma}(W,\theta) = \sigma^{(\perp)} - \sigma^{(\parallel)} = \frac{q}{k} \sum_{n=2}^{2\ell_{\mathrm{max}}} \left(a_{\ell_{\mathrm{max}}}\right)_n^{\check{\Sigma}} (W)\, P_n^2(\cos\theta)$$

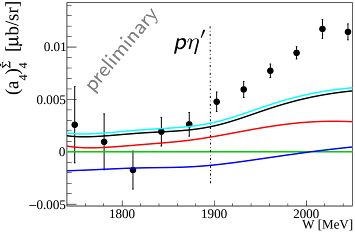
[PhD thesis, F. Afzal (2019)]





$\check{\Sigma}_{\mathrm{CBELSA}}$-data in $\gamma p \longrightarrow \eta p$: $p\eta'$-cusp

Consider the Leg.-moment $(a_4)_4^{\check{\Sigma}}$ belonging to the modulation $(a_4)_4^{\check{\Sigma}}\, P_4^2\,(\cos\theta)$:

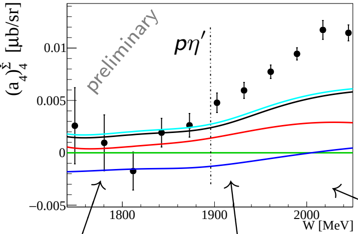


$$(a_4)_4^{\check{\Sigma}} = \langle D, D\rangle + \langle P, F\rangle + \langle F, F\rangle + \langle S, G\rangle + \langle D, G\rangle + \langle G, G\rangle + \langle P, H\rangle + \langle F, H\rangle + \langle H, H\rangle$$

multipoles: BnGa 2014_02

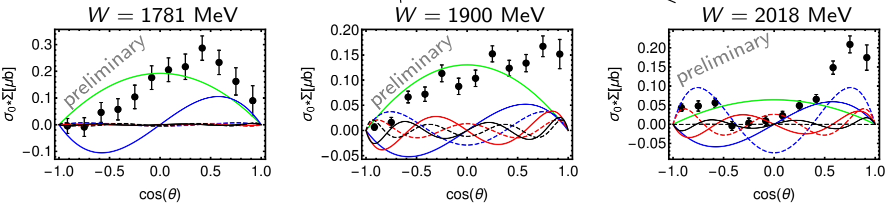
$\check{\Sigma}_{\mathrm{CBELSA}}$-data in $\gamma p \longrightarrow \eta p$: $p\eta'$-cusp

Consider the Leg.-moment $(a_4)_4^{\check{\Sigma}}$ belonging to the modulation $(a_4)_4^{\check{\Sigma}} P_4^2(\cos\theta)$:



$$
\begin{aligned}
(a_4)_4^{\check{\Sigma}} = {} & \langle D, D\rangle + \langle P, F\rangle \\
& + \langle F, F\rangle + \langle S, G\rangle \\
& + \langle D, G\rangle + \langle G, G\rangle \\
& + \langle P, H\rangle + \langle F, H\rangle \\
& + \langle H, H\rangle
\end{aligned}
$$

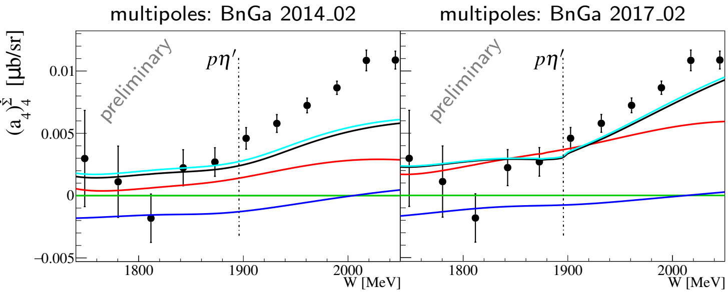
multipoles: BnGa 2014_02



Green: $(a_4)_2^{\check{\Sigma}} P_2^2(\cos\theta)$, Blue: $(a_4)_3^{\check{\Sigma}} P_3^2(\cos\theta)$, Blue-dashed: $(a_4)_4^{\check{\Sigma}} P_4^2(\cos\theta)$,
Red: $(a_4)_5^{\check{\Sigma}} P_5^2(\cos\theta)$, Red-dashed: $(a_4)_6^{\check{\Sigma}} P_6^2(\cos\theta)$, ...

$\check{\Sigma}_{\mathrm{CBELSA}}$-data in $\gamma p \longrightarrow \eta p$: $p\eta'$-cusp

Consider the Leg.-moment $(a_4)_4^{\check{\Sigma}}$ for two different BnGa-solutions:



$$(a_4)_4^{\check{\Sigma}} = \langle D, D \rangle + \langle P, F \rangle + \langle F, F \rangle + \langle \boldsymbol{S}, \boldsymbol{G} \rangle + \langle D, G \rangle + \langle G, G \rangle + \langle P, H \rangle + \langle F, H \rangle + \langle H, H \rangle$$

$\hookrightarrow$ $p\eta'$-cusp can become visible in a (small) Legendre-moment of the polarization observable $\check{\Sigma}$ due to:

1. High statistics of this new dataset,
2. Coverage of the full solid angle with good angular resolution!

Conclusions

$*$) Moment analysis is a simple but (quite) effective method to project some information on partial wave contributions out of the data.

$\longrightarrow$ Fit Legendre-coefficients (moments) $a_k^{\alpha}(W)$

$\longrightarrow$ Get reliable estimate for the lower bound of ℓ_{max} out of the data
Careful: (i) high-low partial wave interferences
(ii) systematic errors!

$\longrightarrow$ Replace "bump-hunting" in the data itself by "interference-hunting" in the Legendre-coefficients, in combination with comparisons to models

$*$) In case data are precise enough, moment analysis can also be helpful for the study of non-analyticities due to the opening of thresholds ('cusps').

Thank You!